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# The singular Riemann–Roch theorem and Hilbert–Kunz functions

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**Abstract**

In the paper, via the singular Riemann–Roch theorem, it is proved that the class of the $e$th Frobenius power ${}^eA$ can be described using the class of the canonical module $\omega_A$ for a normal local ring $A$ of positive characteristic. As a corollary, we prove that the coefficient $\beta(I, M)$ of the second term of the Hilbert–Kunz function $\ell_A(M/I^{[p^e]}M)$ of $e$ vanishes if $A$ is a $\mathbb{Q}$-Gorenstein ring and $M$ is a finitely generated $A$-module of finite projective dimension.

For a normal algebraic variety $X$ over a perfect field of positive characteristic, it is proved that the first Chern class of the $e$th Frobenius power $F^e_*\mathcal{O}_X$ can be described using the canonical divisor $K_X$.


## 1. Introduction

Let $(A, \mathfrak{m})$ be a $d$-dimensional Noetherian local ring of characteristic $p$, where $p$ is a prime integer. Here, $\mathfrak{m}$ is the unique maximal ideal of $A$. For an $\mathfrak{m}$-primary ideal $I$ and a positive integer $e$, we set

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$$I^{[p^e]} = \left(a^{p^e} \mid a \in I\right)A.$$

It is easy to see that $I^{[p^e]}$ is an $\mathfrak{m}$-primary ideal of $A$. For a finitely generated $A$-module $M$, the function $\ell_A(M/I^{[p^e]}M)$ of $e$ is called the *Hilbert–Kunz function* of $M$ with respect to $I$, where $\ell_A(\ )$ stands for the length of the given $A$-module. It is known that

$$\lim_{e\to\infty} \frac{\ell_A(M/I^{[p^e]}M)}{p^{de}}$$

exists [9], and this limit is called the *Hilbert–Kunz multiplicity*, which is denoted by $e_{\mathrm{HK}}(I, M)$. Several properties of $e_{\mathrm{HK}}(I, M)$ have been studied by many authors (Monsky, Watanabe, Yoshida, Huneke, Enescu, etc.).

Recently Huneke, McDermott and Monsky [5, Theorems 1 and 1.11, Corollary 1.10] proved the following exciting theorem:

**Theorem 1.1** *(Huneke, McDermott and Monsky). Let $(A, \mathfrak{m})$ be a $d$-dimensional excellent normal local ring of characteristic $p$, where $p$ is a prime integer. Assume that the residue class field of $A$ is perfect.*

*Let $I$ be an $\mathfrak{m}$-primary ideal of $A$ and $M$ be a finitely generated $A$-module.*

(1) *There exists a real number $\beta(I, M)$ that satisfies the following equation:*[2]

$$\ell_A\left(M/I^{[p^e]}M\right) = e_{\mathrm{HK}}(I, M)\cdot p^{de} + \beta(I, M)\cdot p^{(d-1)e} + O\left(p^{(d-2)e}\right).$$

(2) *Assume that $A$ is F-finite.*[3] *Then, there exists a $\mathbb{Q}$-homomorphism $\tau_I : \mathrm{Cl}(A)_{\mathbb{Q}} \to \mathbb{R}$ that satisfies*

$$\beta(I, M) = \tau_I\left(\mathrm{cl}(M) - \frac{\mathrm{rank}_A M}{p^d - p^{d-1}}\,\mathrm{cl}\left({}^1A\right)\right)$$

*for any finitely generated torsion-free $A$-module $M$. In particular, we have*

$$\beta(I, A) = -\frac{1}{p^d - p^{d-1}}\tau_I\left(\mathrm{cl}\left({}^1A\right)\right).$$

We denote by $\mathbb{Q}$ (respectively $\mathbb{R}$) the field of rational numbers (respectively real numbers). For an abelian group $N$, $N_{\mathbb{Q}}$ stands for $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

The map $\mathrm{cl} : \mathrm{G}_0(A) \to \mathrm{Cl}(A)$ is defined by Bourbaki [1] and sometimes called the *determinant map* (see Remark 2.1 below).

It is natural to ask the following questions:

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[2] Let $f(e)$ and $g(e)$ be functions of $e$. We denote $f(e) = O(g(e))$ if there exists a real number $K$ that satisfies $|f(e)| < Kg(e)$ for all $e \gg 0$.

[3] We say that $A$ is F-finite if the Frobenius map $F : A \to A = {}^1A$ is module-finite. We sometimes denote the $e$th iteration of $F$ by $F^e : A \to A = {}^eA$.