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Finite generation of Ext for a generalization of D-Koszul algebras

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Abstract

In this paper we study the finite generation of Ext-algebras of a class of algebras called δ-resolution determined algebras. We characterize the δ-resolution determined algebras which are monomial algebras. If Λ is a graded algebra such that the associated monomial algebra is δ-resolution determined, we classify when the Ext-algebra of Λ is finitely generated.

Keywords: Koszul algebra; Ext-algebra; Monomial algebra

0. Introduction

The main aim of this paper is to understand, for a certain class of graded algebras, when their Ext-algebras are finitely generated. More precisely, let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ be a positively $\mathbb{Z}$-graded K-algebra where K is a field. Furthermore, assume that Λ_0

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is a semisimple K-algebra. The *Ext-algebra of* Λ, $E(\Lambda)$, is defined to be $E(\Lambda) = \bigoplus_{n\geqslant 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$ where the multiplication is given by the Yoneda product.

In general, $E(\Lambda)$ is not finitely generated, even if Λ is, additionally, a finite-dimensional K-algebra. However, there are some classes of graded algebras where the Ext-algebra is known to be finitely generated, including Koszul algebras [3,11], D-Koszul algebras [2,12] and algebras of finite global dimension.

In general, necessary and sufficient conditions on a graded algebra for the Ext-algebra to be finitely generated are far from understood. For monomial algebras, this problem is studied in [5] where conditions are found if the quiver of the algebra is a cycle. The extension of the case from the quiver of the algebra being a cycle to the general monomial case is discussed in [5,15].

If Λ is a (not necessarily graded) finite-dimensional K-algebra, then $E(\Lambda) = \bigoplus_{n\geqslant 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$ where $\Lambda_0 = \Lambda/\mathfrak{a}$ and $\mathfrak{a}$ is the Jacobson radical of Λ. Again, there is no general characterization of when the Ext-algebra $E(\Lambda)$ is finitely generated. Although we do not consider the non-graded case in this paper, it is well known that $E(\Lambda)$ is finitely generated when Λ is a group algebra of a finite group [6,16].

For a positively $\mathbb{Z}$-graded K-algebra $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ where K is a field and Λ_0 is a finite-dimensional semisimple K-algebra, the graded Jacobson radical of Λ is $\Lambda_1 \oplus \Lambda_2 \oplus \cdots$, which we denote by $\mathfrak{r}$. In this setting, there are minimal graded projective Λ-resolutions of graded Λ-modules. In particular, viewing Λ_0 as a graded right Λ-module, there is a minimal graded projective Λ-resolution of Λ_0

$$(P^*, d^*): \cdots \to P^n \xrightarrow{d^n} P^{n-1} \xrightarrow{d^{n-1}} \cdots \xrightarrow{d^1} P^0 \to \Lambda_0 \to 0.$$

Recall that this resolution is minimal if $d^n(P^n) \subseteq P^{n-1}\mathfrak{r}$ for all $n \geqslant 1$. Denote the global dimension of Λ by $\operatorname{gldim} \Lambda$.

In [10], such an algebra Λ is defined to be *δ-resolution determined* if there is a map $\delta : \mathbb{N} \to \mathbb{N}$ such that, for all n with $n \leqslant \operatorname{gldim} \Lambda$, the projective module P^n in a minimal graded projective Λ-resolution of Λ_0 is generated in degree $\delta(n)$. In addition, Λ is *δ-Koszul* if Λ is δ-resolution determined and if $E(\Lambda)$ is a finitely generated K-algebra.

The class of δ-resolution determined algebras includes Koszul algebras and D-Koszul algebras. In [4], Brenner, Butler and King define a left (p, q)-Koszul ring which turns out to be a δ-resolution determined ring where $\delta(n) = \alpha p + \beta$ for $n = \alpha q + \beta$ with $0 \leqslant \beta < q$. Koszul algebras and D-Koszul algebras are in fact δ-Koszul. The left (p, q)-Koszul rings of [4] are also δ-Koszul.

Our paper begins with some background material on the structure of the Ext-algebra of a monomial algebra. In Section 2, we then study δ-resolution determined monomial algebras, introducing a new class of monomial algebras which we call (D, A, B)-stacked monomial algebras. We prove that the δ-resolution determined monomial algebras are precisely the (D, A, B)-stacked monomial algebras. We note that the (D, A, B)-stacked monomial algebras with $B = 0$ are the (D, A)-stacked monomial algebras of [13].

In Section 3, we briefly review the Gröbner basis theory for path algebras $K\mathcal{Q}$. The definitions of the length-lexicographic order and a reduced Gröbner basis can be found there. Let J be the ideal of $K\mathcal{Q}$ generated by the arrows. Suppose that $\Lambda = K\mathcal{Q}/I$, where I is an ideal of $K\mathcal{Q}$ contained in the ideal J^2. We denote the *length* of a path $p \in K\mathcal{Q}$ by