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Journal of Algebra 295 (2006) 543–561
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On normal subgroups and height zero Brauer characters in a p-solvable group

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Received 16 December 2004

Available online 17 March 2005

Communicated by Gordon James

1. Introduction

Fix a prime number p and let G be a p-solvable finite group. Next, let B be a p-block of G with defect group D and call $\widetilde{B}$, the Brauer correspondent of B in $N_G(D)$. Then, with [2, Theorem 2.1] in mind, Okuyama [15, Theorem 4.1] has shown that B and $\widetilde{B}$ contain equal numbers of irreducible Brauer characters of height zero.

Now let N be a normal subgroup of G and let b be a p-block of N. Assume B covers b. Following M. Murai [12, Section 2], a defect group Q of B is called an *inertial defect group* of B provided that it is a defect group for the Fong–Reynolds correspondent of B in the inertial group T of b in G.

Let φ be an irreducible Brauer character belonging to b. Write $\mathrm{IBr}(B|\varphi)$ for the set of irreducible Brauer characters in B lying over φ. Also, denote by $\mathrm{IBr}^0(B|\varphi)$, the set of all those elements in $\mathrm{IBr}(B|\varphi)$ of height zero. In view of [12, Theorem 4.4(ii)], $\mathrm{IBr}^0(B|\varphi) \neq \emptyset$ if and only if φ is of height zero and is Q-stable for some inertial defect group Q of B.

The main purpose of this paper is to prove the following generalization of the above mentioned theorem of Okuyama.

Theorem. *Let $N \lhd G$, where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Call T the inertial group of b in G and let D be an inertial defect group of B. Let φ be any irreducible Brauer character belonging to b. If $\widehat{B}$*

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.01.045

is the p-block of $NN_G(D)$ with defect group D such that $\widehat{B}^G = B$, then $|\mathrm{IBr}^0(B|\varphi)| = |\bigcup_{t\in T} \mathrm{IBr}^0(\widehat{B}|\varphi^t)|$.

2. Relative π-blocks

One important ingredient we will use to prove the main theorem is the concept of a relative p-block as defined in [8,9]. The present section serves two purposes. First we give a brief review of the notion of a relative p-block and related concepts, and then prove a few results on relative p-blocks needed in the next section. For the sake of generality, instead of working with a single prime, we will work instead with a set of prime numbers.

Throughout this section we fix a set π of primes and a π-separable finite group G. Following Isaacs [6, Section 5], the restriction χ^0 of an ordinary character χ of G to the set of π'-elements of G is called a π'-partial character of G. Moreover, χ^0 is said to be irreducible if it cannot be written as a sum of two π'-partial characters and we write $\mathrm{I}_{\pi'}(G)$ for the set of irreducible π'-partial characters of G.

In the case where π consists of a single prime p, $\mathrm{I}_{\pi'}(G)$ coincides with the set $\mathrm{IBr}(G)$ of irreducible Brauer characters of G, and more generally the irreducible π'-partial characters behave as π-generalizations of the irreducible Brauer characters. In particular, for any χ in the set $\mathrm{Irr}(G)$ of ordinary irreducible characters of G, there are uniquely determined nonnegative integers $d_{\chi\psi}$ such that $\chi^0 = \sum_\psi d_{\chi\psi}\psi$, where ψ runs through $\mathrm{I}_{\pi'}(G)$.

For $\psi \in \mathrm{I}_{\pi'}(G)$, it is clear that there exists $\chi \in \mathrm{Irr}(G)$ such that $\psi = \chi^0$. However, χ is not uniquely determined in general. Nevertheless, Isaacs [4] has canonically constructed a subset $\mathrm{B}_{\pi'}(G)$ of $\mathrm{Irr}(G)$ such that the restriction map $\chi \mapsto \chi^0$ defines a bijection from $\mathrm{B}_{\pi'}(G)$ onto $\mathrm{I}_{\pi'}(G)$.

For the remainder of this section we let $N \lhd G$ and $\mu \in \mathrm{B}_{\pi'}(N)$. The characters χ, $\chi' \in \mathrm{Irr}(G|\mu)$ (the set of characters in $\mathrm{Irr}(G)$ lying over μ) are said to be linked if there is $\psi \in \mathrm{I}_{\pi'}(G)$ for which $d_{\chi\psi} \neq 0$ and $d_{\chi'\psi} \neq 0$. The equivalence classes defined by the transitive extension of this linking are called relative π-blocks of G with respect to (N, μ), and the set of these relative π-blocks is denoted by $\mathrm{Bl}_\pi(G|\mu)$. (See [9].) Note that the relative π-blocks of G with respect to $(\langle 1\rangle, 1_{\langle 1\rangle})$ are exactly Slattery π-blocks of G as defined in [16]. Also, for any Slattery π-block $\mathcal{B}$ of G satisfying $\mathcal{B} \cap \mathrm{Irr}(G|\mu) \neq \emptyset$, one can easily see that $\mathcal{B} \cap \mathrm{Irr}(G|\mu)$ is a union of some relative π-blocks in $\mathrm{Bl}_\pi(G|\mu)$. We should also mention that a notion of defect groups of a relative π-block in $\mathrm{Bl}_\pi(G|\mu)$ is defined in [9, Section 4] and that this definition agrees with that in [17, Definition 2.2] of defect groups of a Slattery π-block when N is trivial.

Let $B \in \mathrm{Bl}_\pi(G|\mu)$. We write $\mathrm{I}_{\pi'}(B)$ for the set of π'-partial characters $\psi \in \mathrm{I}_{\pi'}(G)$ of the form $\psi = \chi^0$ where $\chi \in B$. Then it is obvious that $\mathrm{I}_{\pi'}(B) \subseteq \mathrm{I}_{\pi'}(G|\mu^0)$, the set of $\phi \in \mathrm{I}_{\pi'}(G)$ for which μ^0 is a constituent of ϕ_N. Also, if $\theta \in B$, then there is $\lambda \in \mathrm{B}_{\pi'}(G)$ with $d_{\theta\lambda^0} \neq 0$. By [9, Lemma 3.3], $\lambda \in \mathrm{Irr}(G|\mu)$ and hence $\lambda \in B$, as $d_{\lambda\lambda^0} = 1 \neq 0$. Therefore $\lambda^0 \in \mathrm{I}_{\pi'}(B)$ and consequently $\mathrm{I}_{\pi'}(B) \neq \emptyset$.

Let ω be a π'-partial character of some subgroup H of G. Then the induced object ω^G can be defined using the usual formula for induced characters, but applying it only to π'-elements. For any character α of H with $\alpha^0 = \omega$, we have $(\alpha^0)^G = (\alpha^G)^0$ and hence ω^G is a π'-partial character of G.