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A geometric approach to the Jacobian conjecture in dimension two

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Abstract

The Jacobian conjecture in dimension two holds true if a conjecture, which we call below the *generalized Sard property* (GSP), holds true for the affine plane and an $\mathbb{A}^1$-fibration on $\mathbb{A}^2$. The observations will be made for affine pseudo-planes with the Jacobian conjecture enlarged to the generalized Jacobian conjecture.

Keywords: Jacobian conjecture; Étale endomorphism; Fibration by curves; One place at infinity

0. Introduction

Let X be a smooth affine surface defined over the complex field $\mathbb{C}$. We consider the following conjecture (cf. [4]).

Generalized Jacobian conjecture. *Let $\varphi : X \to X$ be an étale endomorphism. Then φ is a finite morphism.*

Particularly interesting of the conjecture is the case where X is an affine pseudo-plane. Namely, X has an $\mathbb{A}^1$-fibration $\rho : X \to B$ such that B is isomorphic to $\mathbb{A}^1$ and every fiber is irreducible and reduced possibly except for a single irreducible multiple fiber dF_0 with $d \geqslant 1$ and $F_0 \cong \mathbb{A}^1$. For the definition of $\mathbb{A}^1$-fibration and relevant results, see [8]. Hence the affine

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plane $\mathbb{A}^2$ is an affine pseudo-plane as well as the complement of a projective plane curve defined by an equation $X_0^{d-1}X_1 = X_2^d$ with $d > 1$.

It is known that if X is not an affine pseudo-plane then the generalized Jacobian conjecture does not necessarily hold. See [2,4,6,9] for counterexamples and relevant results. In the present article, we shall show that the generalized Jacobian conjecture for affine pseudo-planes is reduced to the validity of the following conjecture.

Generalized Sard property (GSP). *Let X be a smooth affine surface with trivial canonical divisor K_X. Let $\rho : X \to B$ be a fibration of curves whose general fiber has one place at infinity, where B is a smooth algebraic curve, and let $\varphi : X \to X$ be an étale endomorphism such that* $\operatorname{codim}_X(X - \varphi(X)) \geqslant 2$. *Then the image $\varphi(F)$ of a general fiber F is a smooth curve.*

The canonical divisor K_X is not trivial for an affine pseudo-plane in general. In this case, we let $\tilde{X}$ be the universal covering of X, which is a cyclic covering with Galois group $\mathbb{Z}/d\mathbb{Z}$ and obtained as the normalization of $(X, \rho) \times_B (\tilde{B}, \sigma)$, where $\sigma : \tilde{B} \to B$ is a cyclic covering of order d ramifying totally over the point $\rho(F_0)$ and the point at infinity (see [7]). Then one can show that $K_{\tilde{X}} \sim 0$, the étale endomorphism φ lifts to an étale endomorphism $\tilde{\varphi}$ of $\tilde{X}$ with $\operatorname{codim}_{\tilde{X}}(\tilde{X} - \tilde{\varphi}(\tilde{X})) \geqslant 2$ and the fibration ρ lifts to an $\mathbb{A}^1$-fibration $\tilde{\rho} : \tilde{X} \to \tilde{B}$ satisfying $\rho \cdot \mu = \sigma \cdot \tilde{\rho}$, where $\mu : \tilde{X} \to X$ is the covering morphism. Thus one can consider the (GSP) for a pair $(\tilde{\varphi}, \tilde{\rho})$ anew. If it holds, then $\tilde{\varphi}$ becomes an automorphism by Lemma 2.1, which implies, in turn, that φ is an automorphism.

The triviality of K_X will contribute to the smoothness of the image curve $\varphi(F)$ at infinity when a general fiber F of ρ has geometric genus 0 or 1 (see Theorem 1.5). If the (GSP) is valid, it follows that $\varphi(F)$ itself is smooth and moves in a linear pencil without base points.

Although there are no significant results which confirm the validity of the (GSP), it seems important in studying étale endomorphisms. We shall give several observations in the third section including the examples due to R.V. Gurjar which shows that the (GSP) does not necessarily hold if the étaleness condition is dropped.

1. Fibrations of curves and étale endomorphisms

In this section, let X be a smooth affine surface defined over $\mathbb{C}$ and let $\varphi : X \to X$ be an étale endomorphism. We write φ as $\varphi : X_1 \to X_2$ to distinguish the source X from the target X, where $X_1 \cong X_2 \cong X$. We suppose that X has a fibration of curves $\rho : X \to B$, whose general fibers are irreducible and reduced by definition. We assume that B is a rational curve and that *a general fiber C of ρ is a curve of genus $g \geqslant 0$ and has only one place at infinity.* When we consider ρ on X_1, we denote it by $\rho_1 : X_1 \to B_1$. We also assume that $\operatorname{codim}_{X_2}(X_2 - \varphi(X_1)) \geqslant 2$. Namely, the image $\varphi(X_1)$ misses at most finitely many points of X_2. This assumption is satisfied if $\operatorname{Pic}(X)$ is a torsion group and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ [7, Lemma 1.1]. Hence an affine pseudo-plane satisfies this assumption. Mostly for the technical reasons, *we assume that X is a rational surface with* $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$.

Lemma 1.1. *There exist smooth normal completions V_1, V_2 of X_1, X_2 and a morphism $\Phi : V_1 \to V_2$ such that the following conditions are satisfied.*

(1) *The boundary divisor $D_i := V_i - X_i$ $(i = 1, 2)$ is a divisor with simple normal crossings.*