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Separable subsets of GFERF negatively curved groups ☆

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Abstract

A word hyperbolic group G is called GFERF if every quasiconvex subgroup coincides with the intersection of finite index subgroups containing it. We show that in any such group, the product of finitely many quasiconvex subgroups is closed in the profinite topology on G.

Keywords: Word hyperbolic groups; Profinite topology; GFERF

1. Introduction

Let G be a finitely generated group. The *profinite topology* $\mathcal{PT}(G)$ on G is defined by proclaiming all finite index normal subgroups to be the basis of open neighborhoods of the identity element. It is easy to see that G equipped with this topology becomes a topological group. This topology is Hausdorff if and only if G is residually finite.

A subset $P \subseteq G$ will be called *separable* if it is closed in the profinite topology on G. Thus, a subgroup $H \leqslant G$ is separable whenever it is an intersection of finite index subgroups. The group G is said to be *locally extended residually finite* (LERF) if every finitely generated subgroup $H \leqslant G$ is separable.

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A famous theorem of M. Hall states that free groups are LERF. Among other well-known examples of LERF groups are surface groups and fundamental groups of compact Seifert fibred 3-manifolds [26]. In [25] P. Schupp provided certain sufficient conditions for a Coxeter group to be LERF. More recently, R. Gitik [7] constructed an infinite family of LERF hyperbolic groups that are fundamental groups of hyperbolic 3-manifolds.

In 1991 Pin and Reutenauer [23] conjectured that a product of finitely many finitely generated subgroups in a free group is separable and listed some possible applications to groups and semi-groups. In 1993 Ribes and Zalesskiĭ [24] showed that the statement of this conjecture is true. Later a similar question was studied in other LERF groups by Coulbois [5], Gitik [8], Niblo [21], Steinberg [28] and others.

In particular, Gitik in [8, Theorem 1] proved that in a LERF hyperbolic group, a product of two quasiconvex subgroups, one of which is malnormal, is separable.

However, many word hyperbolic groups are not LERF. For example, an ascending HNN-extension of a finite rank free group is never LERF but very often hyperbolic (see [13]). So, it makes sense to use the weaker notion below.

We will say that a (word) hyperbolic group G is GFERF if every quasiconvex subgroup $H \leqslant G$ is separable. The definition of a GFERF Kleinian group Γ was given by Long and Reid in [17]: Γ is called *geometrically finite extended residually finite* (GFERF) if each geometrically finite subgroup $H \leqslant \Gamma$ is separable. Our definition is in the same spirit because in any word hyperbolic group (more generally, in any automatic group) a subgroup is geometrically finite if and only if it is quasiconvex (see [29]).

Long, Reid and Agol gave several examples of GFERF groups [2,17,18]. Hsu and Wise [12] proved that certain right-angled Artin groups are GFERF. Some negatively curved (i.e., word hyperbolic) groups with this property were studied by Gitik in [7]. In the paper [31] Wise provided another large family of GFERF hyperbolic groups; he also showed that Figure 8 knot group is GFERF. The fact that this group is LERF follows from the recent proofs by Agol [1] and Galegari and Gabai [4] of Marden's "tameness" conjecture. This conjecture provides a new way for obtaining LERF and GFERF groups as fundamental groups of 3-manifolds.

The main goal of this paper is to prove the following

Theorem 1.1. *Assume G is a GFERF word hyperbolic group, $G_1, G_2, \dots, G_s$ are quasiconvex subgroups, $s \in \mathbb{N}$. Then the product $G_1 G_2 \cdots G_s$ is separable in G.*

Since a finitely generated subgroup of a finite rank free group is quasiconvex, the above theorem generalizes the result of Ribes and Zalesskiĭ [24] and provides an alternative proof of the conjecture [23]. An application of Theorem 1.1 to the case when $s = 2$ and G_2 is malnormal gives the statement of Gitik's theorem [8, Theorem 1].

Our proof of Theorem 1.1 uses geometry of quasigeodesics in negatively curved spaces and basic properties of quasiconvex subgroups.

A subgroup H of a group G will be called *almost malnormal* if for every $x \in G \setminus H$ the intersection $H \cap xHx^{-1}$ is finite. H is said to be *elementary* if it is virtually cyclic. It is well known that in a hyperbolic group G any element of infinite order belongs to a unique *maximal elementary subgroup*. Thus, any maximal elementary subgroup of G is almost malnormal.

A famous open problem in Geometric Group Theory addresses the existence of a (word) hyperbolic group that is not residually finite. The author would like to emphasize the importance of studying GFERF hyperbolic groups through the proposition below.