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On intersections of orbital varieties and components of Springer fiber

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Abstract

We consider Springer fibers and orbital varieties for GL_n. We show that the irreducible components of an intersection of components of Springer fiber are in bijection with the irreducible components of intersection of orbital varieties; moreover, the corresponding irreducible components in this correspondence have the same codimension. Finally we give a sufficient condition to have an intersection in codimension one.

Keywords: Flag manifold; Springer fibers; Orbital varieties; Robinson–Schensted correspondence; Schubert cell

1. Introduction

1.1. Let G be a semisimple (connected) complex algebraic group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ on which G acts by the adjoint action. For $g \in G$ and $u \in \mathfrak{g}$ we denote this action by $g.u := gug^{-1}$.

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Fix a Cartan subalgebra $\mathfrak{h}$. Let $\mathcal{W}$ denote the associated Weyl group. We have the Chevalley–Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,$$

where $\mathcal{R}$ is the root system of $\mathfrak{g}$ relatively to $\mathfrak{h}$. Let Π be a set of simple roots of $\mathcal{R}$. Denote $\mathcal{R}^+$ (respectively $\mathcal{R}^-$) the positive roots (respectively negative roots) (w.r.t. Π). We sometimes prefer the notation $\alpha > 0$ (respectively $\alpha < 0$) to designate a positive (respectively negative) root. Let $\mathfrak{b} := \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha$ be the standard Borel subalgebra (w.r.t. Π) and $\mathfrak{n} := \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha$ its nilpotent radical. Let B be the Borel subgroup of G with $\mathrm{Lie}(B) = \mathfrak{b}$.

Let $G \times^B \mathfrak{n}$ be the space obtained as the quotient of $G \times \mathfrak{n}$ by the right action of B given by $(g, x).b := (gb, b^{-1}.x)$ with $g \in G$, $x \in \mathfrak{n}$ and $b \in B$. By the Killing form we get the following identification $G \times^B \mathfrak{n} \simeq T^*(G/B)$. Let $g * x$ denote the class of (g, x) and $\mathcal{F} := G/B$ the flag manifold. The map $G \times^B \mathfrak{n} \to \mathcal{F} \times \mathfrak{g}$, $g * x \mapsto (gB, g.x)$ is an embedding which identify $G \times^B \mathfrak{n}$ with the following closed subvariety of $\mathcal{F} \times \mathfrak{g}$ (see [16, p. 19]):

$$\mathcal{Y} := \{(gB, x) \mid x \in g.\mathfrak{n}\}.$$

The map $f : G \times^B \mathfrak{n} \to \mathfrak{g}$, $g * x \mapsto g.x$ is called the *Springer resolution* and we have the following commutative diagram:

$$\begin{array}{ccc} G \times^B \mathfrak{n} & \xrightarrow{\ \simeq\ } & \mathcal{Y} \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & \mathfrak{g} & \end{array}$$

where $pr_2 : \mathcal{F} \times \mathfrak{g} \to \mathfrak{g}$, $(gB, x) \mapsto x$. The map f is proper (because G/B is complete) and its image is exactly $G.\mathfrak{n} = \mathcal{N}$, the *nilpotent variety* of $\mathfrak{g}$ [21].

Let x be a nilpotent element in $\mathfrak{n}$. By the diagram above we have:

$$\mathcal{F}_x := f^{-1}(x) = \{gB \in \mathcal{F} \mid x \in g.\mathfrak{n}\} = \{gB \in \mathcal{F} \mid g^{-1}.x \in \mathfrak{n}\}. \tag{1.1}$$

The variety $\mathcal{F}_x$ is called the *Springer fiber* above x and has been studied by many authors. It was one of the most stimulating subjects during the last three decades, appearing in many areas, for example, in representation theory and singularity theory. But it remains a very mysterious object, and the major difficulty is its geometric description which is known in a few cases. For x in the regular orbit in $\mathfrak{g}$ it is reduced to one point. For x in the subregular orbit in $\mathfrak{g}$ it is a finite union of projective lines which intersect themselves transversally and is usually called the *Dynkin curve*, it was obtained by J. Tits (see e.g. [24, Theorem 2, p. 153]). For x in the minimal orbit its irreducible components are some Schubert varieties [2].

The Springer fibers arise in many contexts. They arise as fibers of Springer's resolution of singularities of the nilpotent variety in [16,17,21]. In the course of these investigations,