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Flatness for the moment map for representations of quivers

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Abstract

We study the flatness for the moment map associated to the cotangent bundle of the space of representations of a quiver Q. If the associated moment map of a root is flat, then we call the root a flat root. We first study the flat roots in the fundamental set of a quiver Q. Then we give an explicit description of all the flat roots of Q. This description is obtained by using the natural class of (-1)-reflections, which are introduced in this paper. We also show that there are only a finite number of flat roots in each orbit under the Weyl-group of the quiver Q.

Keywords: Flat root; Moment map; Representation variety

1. Introduction

We study the flatness for the moment map $\mu_{\mathbf{d}}$ associated to the cotangent bundle of the space $\mathrm{Rep}(Q, \mathbf{d})$ of representations of a quiver Q. The study of the moment map goes back to Kronheimer who uses this moment map in his study of Kleinian singularities and their

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deformations for euclidean quivers [7]. The moment map is also important in quantum groups. Lusztig and Kashiwara use the properties of the moment map in the construction of the canonical basis and the crystal basis, in [5,8], respectively, for the negative part of a quantum group. In [1–3] Crawley-Boevey does important work on the geometry of the moment map. In particular he studies the flatness of the moment map in [1]. The main aim of this paper is to give clear and complete description of when the moment map is flat.

By Lemma 4.5 in [1] we know that $\mathbf{d}$ is a Schur root if $\mu_{\mathbf{d}}$ is flat, so we call $\mathbf{d}$ a flat root if $\mu_{\mathbf{d}}$ is flat. The first step in understanding the flatness is to understand the flat roots in the fundamental set $\mathcal{F}$ of a quiver Q.

Theorem 1.1. *Let* $\mathbf{d}$ *be a root in the fundamental set* $\mathcal{F}$ *of a quiver* Q. *Then the moment map* $\mu_{\mathbf{d}}$ *is flat if and only if* $\mathbf{d}$ *is not in the form of* $m\delta$, *where* $m \geqslant 2$ *and* δ *is in* $\mathcal{F}$ *minimal with* $q(\delta) = 0$.

Note that a simple root of a quiver is always flat. Using a special class of reflections in the Weyl group of Q, which we call (-1)-reflections, and Theorem 1.1 we are able to give a complete description of the flat roots of a quiver Q. The definition of (-1)-reflection and (-1)-equivalence in the following will be given in Section 2. We say that a flat root is fundamental if it is a simple root or a flat root in the fundamental set $\mathcal{F}$.

Theorem 1.2. *Let* $\mathbf{d}$ *be a root of* Q *which is not fundamental. Then the moment map* $\mu_{\mathbf{d}}$ *is flat if and only if* $\mathbf{d}$ *is* (-1)*-equivalent to a fundamental flat root. In this case* $\mathbf{d}$ *is indivisible.*

Theorems 1.1 and 1.2 give an effective algorithm for checking if the associated moment map of a given root is flat. By Theorem 1.2 it is not difficult to determine all the flat roots which are (-1)-equivalent to a given flat root. Moreover, we show that the (-1)-equivalent class of a flat root, $\mathcal{W}_f(\mathbf{d}) = \{\sigma(\mathbf{d}) \mid \sigma \in$ the Weyl group $\mathcal{W}$ of Q and $\sigma(\mathbf{d})$ is flat$\}$, is a finite set.

Theorem 1.3. *Let* $\mathbf{d}$ *be a fundamental flat root of* Q. *Then the* (-1)*-equivalent class* $\mathcal{W}_f(\mathbf{d})$ *of* $\mathbf{d}$ *is a finite set.*

We arrange the content of this paper as follows. In Section 2 we recall basic definitions, results and notation. In Section 3 we give some examples on flat roots and prove several properties of them. In Section 4 we prove Theorem 1.1. In Section 5 we consider the MWR-decomposition and further properties of flat roots. In Section 6 we prove another combinatorial characterization of the flatness. We give the proof of Theorem 1.2 in Section 7. Finally in Section 8 we prove Theorem 1.3, that is, we can only use finitely many times of (-1)-reflections on a flat root.

2. Background

2.1. Representation varieties of quivers and the moment map

Let $\mathbf{k}$ be an algebraically closed field. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows and s and t are two maps $Q_1 \to Q_0$ with