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Computing relative abelian kernels of finite monoids

Edite Cordeiro [a], Manuel Delgado [b,*]

[a] *Instituto Politécnico de Bragança, Escola Superior de Tecnologia e Gestão, Campus de Santa Apolónia, 5301-857 Bragança, Portugal*
[b] *Centro de Matemática, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*

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Abstract

Let H be a pseudovariety of abelian groups corresponding to a recursive supernatural number. In this note we explain how a concrete implementation of an algorithm to compute the kernel of a finite monoid relative to H can be achieved. The case of the pseudovariety Ab of all finite abelian groups was already treated by the second author and plays an important role here, where we will be interested in the proper subpseudovarieties of Ab. Our work relies on an algorithm obtained by Steinberg.

Introduction and motivation

The problem of computing kernels of finite monoids goes back to the early seventies and became popular among semigroup theorists through the Rhodes Type II conjecture which proposed an algorithm to compute the kernel of a finite monoid relative to the class G of all finite groups. Proofs of the conjecture were given in independent and deep works by Ash [1] and Ribes and Zalesskiĭ [15]. For an excellent survey on the work done around this conjecture, as well as connections with other topics such as the Malcev product, we refer the reader to [13].

* Corresponding author.
E-mail addresses: emc@ipb.pt (E. Cordeiro), mdelgado@fc.up.pt (M. Delgado).

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The work of Ribes and Zalesskiĭ solves a problem on profinite groups (the product of a finite number of finitely generated subgroups of a free group is closed for the profinite topology of the free group) which in turn, using work of Pin and Reutenauer [14], solves the Type II conjecture. Pin and Reutenauer essentially reduced the problem of determining the kernel of a finite monoid to the problem of determining the closure of a finitely generated subgroup of a free group endowed with the profinite topology. This idea was followed by several authors to compute kernels relative to other classes of groups, considering in these cases relatively free groups endowed with topologies given by the classes under consideration. We can refer to Ribes and Zalesskiĭ [16] for the class of all finite p-groups, the second author [3] for the class Ab of all finite abelian groups, and Steinberg [18] for any class of finite abelian groups closed under the formation of homomorphic images, subgroups and finite direct products. A class of finite groups closed under the formation of homomorphic images, subgroups and finite direct products is called a *pseudovariety of groups*.

Steinberg's paper [18] gives an algorithm, on which this work is based, to compute the kernel of a finite monoid relative to any pseudovariety of abelian groups. Since the problem of the existence of an algorithm for the case of locally finite pseudovarieties (which are pseudovarieties containing, for each finite set A, the free object on A in the variety they generate) is trivial, and Steinberg's paper was mostly dedicated to theoretical results, it emphasizes the cases of non-locally finite pseudovarieties. We are aiming to obtain concrete implementations and therefore even the locally finite case requires some work. Concrete implementations of this kind of algorithms are useful, since calculations (that can not be done by hand due to the time required) often give the necessary intuition to formulate conjectures and may help in the subsequent problem solving. A step towards the concrete implementation in the GAP system [19] for the case of the pseudovariety Ab was given in [4] by the second author who also implemented it using the GAP programming language. This algorithm is presently part of a GAP package in preparation which will probably also contain implementations of the algorithms described in this paper. The usefulness of this software can be inferred from a number of papers whose original motivation came from computations done: we can refer to several joint works by Fernandes and the second author [5–8].

In Section 1 of the present paper we recall a few facts concerning the concept of supernatural number and mention a bijective correspondence between the classes of supernatural numbers and pseudovarieties of abelian groups.

In Section 2 we observe that computing the closure of a subgroup of $\mathbb{Z}^n$ (relative to certain topologies) is feasible without too much work using GAP. Notice that we are aiming to use Steinberg's algorithm to compute relative kernels which, as already observed, uses computing relative closures as an essential ingredient.

Section 3 is dedicated to the computation of the closure of semilinear sets relative to the profinite topology. It is relevant for Section 4.

In Section 4 we recall the definition of the kernel of a finite monoid relative to a pseudovariety of groups. Then we dedicate two subsections to the description of the concrete implementations we are proposing. The cases of pseudovarieties of abelian groups corresponding to infinite supernatural numbers and those of pseudovarieties corresponding to natural numbers are treated separately.

Applications will appear in forthcoming papers by the authors and V.H. Fernandes.