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C^0-estimates and smoothness of solutions to the parabolic equation defined by Kimura operators

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A B S T R A C T

Kimura diffusions serve as a stochastic model for the evolution of gene frequencies in population genetics. Their infinitesimal generator is an elliptic differential operator whose second-order coefficients matrix degenerates on the boundary of the domain. In this article, we consider the inhomogeneous initial-value problem defined by generators of Kimura diffusions, and we establish C^0-estimates, which allows us to prove that solutions to the inhomogeneous initial-value problem are smooth up to the boundary of the domain where the operator degenerates, even when the initial data is only assumed to be continuous.

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1. Introduction

The evolution of gene frequencies is one of the central themes of research in population genetics, and one of the natural ways to model the changes of gene frequencies in a population is through the use of Markov chains and their continuous limits. This line of research was initiated by R. Fisher [12], J. Haldane [13], S. Wright [24], and later extended by M. Kimura [16]. The stochastic processes involved in these works are continuous limits of discrete Markov processes, which are solutions to stochastic differential equations whose infinitesimal generator is a degenerate-elliptic partial differential operator. A rigorous understanding of the regularity of solutions to parabolic equations defined by such operators plays a central role in the study of various probabilistic properties of the associated stochastic models.

A wide extension of the generator of continuous limits of the Wright–Fisher model [12,24,13,16,17,23,10,15] was introduced in the work of C. Epstein and R. Mazzeo [6, 7], where the authors build a suitable Schauder theory to prove existence, uniqueness and optimal regularity of solutions to the inhomogeneous initial-value problem defined by generalized Kimura diffusion operators acting on functions defined on compact manifolds with corners. In our work, we extend the regularity results obtained in [6,7] by proving a priori local Schauder estimates of solutions, in which we control the higher-order Hölder norm of solutions in terms of their supremum norm (Theorem 1.1). This result allows us to prove in Theorem 1.5 that the solutions are smooth up to the portion of the boundary where the operator degenerates, even when the initial data is only assumed to be continuous, as opposed to Hölder continuous in [6,7]. In the sequel, we describe our main results and their applications in more detail.

Let $\mathbb{R}_+ := (0,\infty)$ and $S_{n,m} := \mathbb{R}^n_+ \times \mathbb{R}^m$, where n and m are nonnegative integers such that $n + m \geq 1$. While generalized Kimura diffusion operators act on functions defined on compact manifolds with corners [7, §2], from an analytical point of view and due to the fact that we are interested in the local properties of solutions, in our article, we consider a second-order elliptic differential operator of the form

$$\begin{aligned} Lu = &\sum_{i=1}^n \left(x_i a_{ii}(z) u_{x_i x_i} + b_i(z) u_{x_i}\right) + \sum_{i,j=1}^n x_i x_j \tilde{a}_{ij}(z) u_{x_i x_j} \\ &+ \sum_{i=1}^n \sum_{l=1}^m x_i c_{il}(z) u_{x_i y_l} + \sum_{k,l=1}^m d_{kl}(z) u_{y_k y_l} + \sum_{l=1}^m e_l(z) u_{y_l}, \end{aligned} \tag{1.1}$$