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# Properties of Beurling-type submodules via Agler decompositions

Kelly Bickel [a,1], Constanze Liaw [b,*,2]

[a] *Department of Mathematics, Bucknell University, 360 Olin Science Building, Lewisburg, PA 17837, USA*

[b] *CASPER and Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798, USA*

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ABSTRACT

In this paper, we study operator-theoretic properties of the compressed shift operators $S_{z_1}$ and $S_{z_2}$ on complements of submodules of the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. Specifically, we study Beurling-type submodules – namely submodules of the form $\theta H^2(\mathbb{D}^2)$ for $\theta$ inner – using properties of Agler decompositions of $\theta$ to deduce properties of $S_{z_1}$ and $S_{z_2}$ on model spaces $H^2(\mathbb{D}^2) \ominus \theta H^2(\mathbb{D}^2)$. Results include characterizations (in terms of $\theta$) of when a commutator $[S^*_{z_j}, S_{z_j}]$ has rank $n$ and when subspaces associated to Agler decompositions are reducing for $S_{z_1}$ and $S_{z_2}$. We include several open questions.



* Corresponding author.
*E-mail addresses:* kelly.bickel@bucknell.edu (K. Bickel), Constanze_Liaw@baylor.edu (C. Liaw).

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## 1. Introduction

### 1.1. Motivation

The Hardy space on the disk $H^2(\mathbb{D})$ has played a prominent role in developing both function and operator theory over the past century. Of particular importance are its shift-invariant subspaces, which (as proved by Beurling in [10]) are always of the form $\theta H^2(\mathbb{D})$ for an inner function $\theta$. Indeed, the model theory of Sz.-Nagy–Foias [29] shows that every completely non-unitary, $C_0$ contraction is unitarily equivalent to the compression of multiplication by $z$ on some $\mathcal{K}_\theta \equiv H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$, as long as $\theta$ can be operator-valued.

We are interested in generalizations of one-variable Hardy space theory to the Hardy space over the bidisk $H^2(\mathbb{D}^2)$. Substantial progress in this direction has been made by W. Rudin, R.G. Douglas, M. Gadadhar, R. Yang and many others [14,15,17,21,26,31,32], who often frame the important problems in terms of Hilbert submodules. In our situation, a Hilbert submodule $M$ in $H^2(\mathbb{D}^2)$ is a subspace that is invariant under multiplication by the two independent variables $z_1$ and $z_2$ or equivalently, invariant under the Toeplitz operators $T_{z_1}, T_{z_2}$ [17]. We are interested in Beurling-type submodules, namely those of the form:

$$M \equiv \theta H^2(\mathbb{D}^2),$$

where $\theta$ is inner. As shown by Mandrekar in [25], these submodules are exactly the ones on which $T_{z_1}$ and $T_{z_2}$ are doubly commuting. In analogy with one-variable model theory, given $\mathcal{K}_\theta \equiv H^2(\mathbb{D}^2) \ominus \theta H^2(\mathbb{D}^2)$, we are particularly interested in the compressed shift operators:

$$S_{z_1} \equiv P_\theta T_{z_1}|_{\mathcal{K}_\theta} \quad \text{and} \quad S_{z_2} \equiv P_\theta T_{z_2}|_{\mathcal{K}_\theta},$$

where $P_\theta$ denotes the orthogonal projection onto $\mathcal{K}_\theta$ and $\theta$ is inner. The case of general analytic contractions $\theta$ is quite involved even when we consider functions of only one complex variable. See for example [16,24], which concerns Clark theory in the general situation, and the references therein.

The literature already contains a variety of results concerning commutators of $S_{z_1}$, $S_{z_2}$ and their adjoints, as these operators are crucially related to both $\theta$ and the structure of $\mathcal{K}_\theta$. For example, [17,20,21,31,32] contain interesting results concerning the behaviors of the commutators

$$[S_{z_1}, S_{z_2}^*] \text{ and } [S_{z_1}, S_{z_2}].$$

However, the independent behavior of $S_{z_1}$ or $S_{z_2}$ is not completely understood. Some results exist concerning the essential spectrum of these operators under additional conditions [32], but in general, their operator theoretic properties and connections to $\theta$ are