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Nonlinear elliptic equations and intrinsic potentials of Wolff type ☆

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We give necessary and sufficient conditions for the existence of weak solutions to the model equation

$$-\Delta_p u = \sigma\, u^q \quad \text{on } \mathbb{R}^n,$$

in the case $0 < q < p-1$, where $\sigma \geq 0$ is an arbitrary locally integrable function, or measure, and $\Delta_p u = \operatorname{div}(\nabla u |\nabla u|^{p-2})$ is the p-Laplacian. Sharp global pointwise estimates and regularity properties of solutions are obtained as well. As a consequence, we characterize the solvability of the equation

$$-\Delta_p v = b\,\frac{|\nabla v|^p}{v} + \sigma \quad \text{on } \mathbb{R}^n,$$

where $b > 0$. These results are new even in the classical case $p = 2$.

Our approach is based on the use of special nonlinear potentials of Wolff type adapted for "sublinear" problems, and related integral inequalities. It allows us to treat simultaneously several problems of this type, such as equations with general quasilinear operators $\operatorname{div} \mathcal{A}(x, \nabla u)$, fractional Laplacians $(-\Delta)^\alpha$, or fully nonlinear k-Hessian operators.

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1. Introduction

In the present paper, we study elliptic equations of the type

$$\begin{cases} -\Delta_p u = \sigma\, u^q & \text{in } \mathbb{R}^n, \\ \liminf\limits_{x\to\infty} u(x) = 0, \quad u > 0, \end{cases} \tag{1.1}$$

where $0 < q < p-1$, $\Delta_p = \operatorname{div}(\nabla u|\nabla u|^{p-2})$ is the p-Laplace operator, and $\sigma \geq 0$ is an arbitrary locally integrable function, or locally finite Borel measure, $\sigma \in M^+(\mathbb{R}^n)$; if $\sigma \in L^1_{\rm loc}(\mathbb{R}^n)$ we write $d\sigma = \sigma\, dx$.

Our main goal is to give necessary and sufficient conditions on σ for the existence of weak solutions to (1.1), understood in an appropriate renormalized sense. We also obtain matching upper and lower global pointwise bounds, and provide sharp $W^{1,p}_{\rm loc}$-estimates of solutions. On our way, we identify key integral inequalities, and construct new nonlinear potentials of Wolff type that are intrinsic to a number of similar problems.

In particular, our approach is applicable to general quasilinear $\mathcal{A}$-Laplace operators $\operatorname{div} \mathcal{A}(x, \nabla u)$, and fully nonlinear k-Hessian operators, as well as equations with the fractional Laplacian,

$$\begin{cases} (-\Delta)^\alpha u = \sigma\, u^q & \text{in } \mathbb{R}^n, \\ \liminf\limits_{x\to\infty} u(x) = 0, \quad u > 0, \end{cases} \tag{1.2}$$

for $0 < q < 1$ and $0 < \alpha < \frac{n}{2}$; this includes the range $\alpha > 1$ where the usual maximum principle is not available.

In the classical case $p = 2$, equation (1.1), or equivalently (1.2) with $\alpha = 1$, and $0 < q < 1$, serves as a model of *sublinear* elliptic problem. It is easy to see that it is equivalent to the integral equation $u = \mathbf{N}(u^q d\sigma)$, where $\mathbf{N}\omega = (-\Delta)^{-1}\omega$ is the Newtonian potential of $d\omega = u^q d\sigma$ on $\mathbb{R}^n$.

As we emphasize below, equation (1.1) with $p = 2$ and $0 < q < 1$ is governed by the important integral inequality

$$\left(\int_{\mathbb{R}^n} |\varphi|^q \, d\sigma \right)^{\frac{1}{q}} \leq \varkappa \, ||\Delta \varphi||_{L^1(\mathbb{R}^n)}, \tag{1.3}$$

for all test functions $\varphi \in C^2(\mathbb{R}^n)$ vanishing at infinity such that $-\Delta\varphi \geq 0$.

Inequality (1.3) represents the end-point case of the well-studied (L^p, L^q) trace inequalities for $p > 1$. A comprehensive treatment of trace inequalities can be found in [25].

More precisely, we will use a localized version of (1.3) where the measure σ is restricted to a ball $B = B(x, r)$, and the corresponding best constant $\varkappa$ is denoted by $\varkappa(B)$. These constants are used as building blocks in our key tool, a nonlinear potential of Wolff type,