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Blow-up analysis for a cosmic strings equation

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ABSTRACT

In this paper we develop a blow-up analysis for solutions of an elliptic PDE of Liouville type over the plane. Such solutions describe the behavior of cosmic strings (parallel in a given direction) for a W-boson model coupled with Einstein's equation. We show how the blow-up behavior of the solutions is characterized, according to the physical parameters involved, by new and surprising phenomena. For example in some cases, after a suitable scaling, the blow-up profile of the solution is described in terms of an equations that bares a geometrical meaning in the context of the "uniformization" of the Riemann sphere with conical singularities.

1. Introduction

In this paper, we focus on the blow-up analysis for solutions of a Liouville type equation describing the behavior of selfgravitating cosmic strings for a massive W-boson model coupled with Einstein's equation, introduced in [1]. More precisely, for the model

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in [1], it is possible to use Einstein's equation together with a suitable ansatz, in order to reduce the analysis of the corresponding selfgravitating cosmic string located at the origin and parallel to the x_3-direction to the study of the following elliptic problem:

$$\begin{cases} -\Delta u = e^{au} + |x|^{2N} e^{u} \quad \text{in } \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} \left(e^{au} + |x|^{2N} e^{u}\right) < \infty, \end{cases} \tag{1.1}$$

with $a > 0$ a physical parameter and $N \in \mathbb{N}$ the string's multiplicity, see [1,55] and [44] for details.

For a solution u of (1.1) the value

$$\beta := \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(e^{au} + |x|^{2N} e^{u}\right) \tag{1.2}$$

relates to the (finite) string's energy, and our main concern will be to identify the (sharp) range of β's for which problem (1.1)–(1.2) is solvable. We mention that some existence results concerning (1.1) are contained in [9,10,55] and [11].

Here, we shall build our investigation upon the work in [44,45], where the authors characterize completely the radial solvability of (1.1)–(1.2).

To be more specific we observe that, (as shown in [11] and [22]), every solution u of (1.1) and (1.2) satisfies:

$$-\beta \log(|x|+1) - C \leq u(x) \leq -\beta \log(|x|+1) + C, \tag{1.3}$$

$$r\partial_r u \to -\beta \quad , \quad \partial_\vartheta u \to 0 \quad \text{as } r \to +\infty, \tag{1.4}$$

with suitable $C > 0$ (depending on u), and (r, ϑ) the polar coordinates in $\mathbb{R}^2$. Thus, from (1.3) we find that,

$$\beta = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(e^{au} + |x|^{2N} e^{u}\right) > \max\left\{\frac{2}{a}, 2(N+1)\right\}. \tag{1.5}$$

The two values on the right hand side of (1.5) coincide for $a = \dfrac{1}{N+1}$, and in this case, the equation in (1.1) acquires the following scaling invariance property:

$$u(x) \to u_\lambda(x) := u(\lambda x) + 2(N+1)\log\lambda = u(\lambda x) + \frac{2}{a}\log\lambda, \tag{1.6}$$

for every $\lambda > 0$.

In turn, as shown in [11], one can use a Pohozaev's type inequality (in the usual way) to find that, for problem (1.1) the following holds: