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On the non-commutative fractional Wishart process

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A B S T R A C T

We investigate the process of eigenvalues of a fractional Wishart process defined by $N = B^*B$, where B is the matrix fractional Brownian motion recently studied in [18]. Using stochastic calculus with respect to the Young integral we show that, with probability one, the eigenvalues do not collide at any time. When the matrix process B has entries given by independent fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$, we derive a stochastic differential equation in the Malliavin calculus sense for the eigenvalues of the corresponding fractional Wishart process. Finally, a functional limit theorem for the empirical measure-valued process of eigenvalues of a fractional Wishart process is obtained. The limit is characterized and referred to as the *non-commutative fractional Wishart process*, which constitutes the family of fractional dilations of the free Poisson distribution.

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1. Introduction

In this paper, we make a systematic study of the dynamics and the limiting non-commutative distribution of the eigenvalue process of a fractional Wishart matrix process. More specifically, let $H \in (0,1), n, p \geq 1$ and $B = \{\{b_{ij}(t), t \geq 0\}, 1 \leq i \leq p, 1 \leq j \leq n\}$ be a set of $p \times n$ independent one-dimensional fractional Brownian motions with the same Hurst parameter H. That is, each b_{ij} is a zero mean Gaussian process with covariance

$$\mathbb{E}\Big[b_{ij}(t)b_{ij}(s)\Big] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

As in [18], we introduce $(N(t), t \geq 0)$, the matrix fractional Brownian motion process with parameter H whose components satisfy $N_{ij}(t) = b_{ij}(t)$, for $t \geq 0$.

A fractional Wishart process is the nonnegative definite $n \times n$ matrix process defined by $X = N^*N$, where N^* denotes the transpose of some matrix N. Let $(\lambda_1(t), \lambda_2(t), .., \lambda_n(t), t \geq 0)$ be the n-dimensional stochastic process of eigenvalues of X and consider the empirical spectral process of the eigenvalues $\lambda_1^{(n)}(t) \geq \lambda_2^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t) \geq 0$ of $X^{(n)} = n^{-1}X$, i.e.,

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \qquad t \geq 0. \tag{1.1}$$

Different aspects of the dynamics and asymptotics of this spectral process have been considered by several authors in the case $H = 1/2$ of the classical Wishart process. In this case, for $n \geq 1$ fixed, Bru [3] considered the dynamics and non-colliding phenomena of the eigenvalue process, proving that the spectral process is an n-dimensional diffusion given by the system of non-smooth diffusion equations

$$\lambda_i(t) = \lambda_i(0) + 2\int_0^t \sqrt{\lambda_i(s)} \cdot d\nu^i(s) + \int_0^t \left(p + \sum_{i \neq j} \frac{\lambda_i(s) + \lambda_j(s)}{\lambda_i(s) - \lambda_j(s)}\right) ds, \tag{1.2}$$

where ν^i are independent Brownian motions for $i = 1, \ldots, n$, and "$\cdot$" denotes the Itô stochastic integral. Moreover, Bru [3] also showed that if $\lambda_1^{(n)}(0) \geq \cdots \geq \lambda_n^{(n)}(0)$, then a.s. the eigenvalues do not collide at any time, i.e.,

$$\mathbb{P}(\lambda_1(t) > \cdots > \lambda_n(t), \forall t > 0) = 1. \tag{1.3}$$

The main tool for proving (1.2) is Itô's formula for matrix-valued semimartingales, and for (1.3), a McKean type argument in the classical stochastic calculus.

Still in the classical case $H = 1/2$, for fixed $t > 0$, the asymptotic distribution of $\mu_t^{(n)}$ is given by the classical pioneering work of Marchenko and Pastur, [13]. Namely, recall