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Malliavin calculus for regularity structures: The case of gPAM

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A B S T R A C T

Malliavin calculus is implemented in the context of Hairer (2014) [16]. This involves some constructions of independent interest, notably an extension of the structure which accommodates a robust, and purely deterministic, translation operator, in L^2-directions, between “models”. In the concrete context of the generalized parabolic Anderson model in 2D – one of the singular SPDEs discussed in the afore-mentioned article – we establish existence of a density at positive times.

Contents

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1. Introduction

Malliavin calculus [23] is a classical tool for the analysis of stochastic (partial) differential equations, e.g. [26,28] and the references therein. The aim of this paper is to explore Malliavin calculus in the context of Hairer's *regularity structures* [16], a theory designed to provide a solution theory for certain ill-posed stochastic partial differential equations (SPDEs) typically driven by Gaussian (white) noise. By now, there is an impressive list of such equations that can be handled in this framework, many well-known from the (non-rigorous) physics literature: KPZ, parabolic Anderson model, stochastic quantization equation, stochastic Navier–Stokes, ...

At this moment, and despite a body of general results and a general démarche, each equation still needs some tailor-made analysis, especially when it comes to renormalization [16, Sec. 8,9] and convergence of renormalized approximations [16, Sec. 10], in the context of Gaussian white noise. For this reason, we focus on one standard example of the theory – the generalized parabolic Anderson model (gPAM) – although an effort is made throughout, with regard to future adaptions to other equations, to emphasize the main governing principles of our results. To be specific, recall that gPAM is given (formally!) by the following non-linear SPDE

$$(\partial_t - \Delta)u = g(u)\xi, \qquad u(0,\cdot) = u_0(\cdot), \tag{1.1}$$

for $t \geq 0$, g sufficiently smooth, spatial white noise $\xi = \xi(x,\omega)$ and fixed initial data u_0. Assuming periodic boundary conditions, write $x \in \mathbb{T}^d$, the d-dimensional torus. Now a.s. the noise is a Gaussian random distribution, of Hölder regularity $\alpha < -d/2$. Standard reasoning suggests that u (and hence $g(u)$) has regularity $\alpha+2$, due to the regularization of the heat-flow. But the product of two such Hölder distributions is only well-defined, if the sum of the regularities is strictly positive – which is the case in dimension $d = 1$