Journal of Functional Analysis 271 (2016) 3403–3422

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

On reduction theory and Brown measure for closed unbounded operators

K. Dykema [a,*,1], J. Noles [a,1], F. Sukochev [b,2], D. Zanin [b,2]

[a] *Department of Mathematics, Texas A&M University, College Station, TX, USA*
[b] *School of Mathematics and Statistics, University of New South Wales, Kensington, NSW, Australia*

A R T I C L E   I N F O

Article history:
Received 13 September 2015
Accepted 17 September 2016
Available online 21 September 2016
Communicated by D. Voiculescu

MSC:
primary 47C99
secondary 47C15

Keywords:
Direct integral
Brown measure

A B S T R A C T

The theory of direct integral decompositions of both bounded and unbounded operators is further developed; in particular, results about spectral projections, functional calculus and affiliation to von Neumann algebras are proved. For operators belonging to or affiliated to a tracial von Neumann algebra that is a direct integral von Neumann algebra, the Brown measure is shown to be given by the corresponding integral of Brown measures.

1. Introduction

Reduction theory is a way of decomposing von Neumann algebras as direct integrals (a generalization of direct sums) of other von Neumann algebras. It is commonly employed, when the direct integral decomposition is done over the center of the

* Corresponding author.
E-mail addresses: ken.dykema@math.tamu.edu (K. Dykema), jnoles@math.tamu.edu (J. Noles), f.sukochev@math.unsw.edu.au (F. Sukochev), d.zanin@math.unsw.edu.au (D. Zanin).

[1] Research supported in part by NSF grant DMS-1202660.
[2] Research supported by ARC (DP150100920).

http://dx.doi.org/10.1016/j.jfa.2016.09.015

von Neumann algebra, to see that an arbitrary von Neumann algebra is a direct integral of factors. However, the direct integral decomposition can be done over any von Neumann subalgebra of the center.

Our main goal in this paper is to show that, in the context of tracial von Neumann algebras and certain unbounded operators affiliated to such von Neumann algebras, the Brown spectral distribution measure behaves well with respect to direct integral decompositions. This result (Theorem 5.6) is a natural development and its proof is technically nontrivial. This result finds immediate application in the paper [6], that extends results from [5] about Schur upper-triangular forms to certain unbounded operators affiliated to finite von Neumann algebras.

We will now describe some of the theory of Brown measure and the Fuglede–Kadison determinant, on which it depends. Given a tracial von Neumann algebra $(\mathcal{M}, \tau)$, by which we mean a von Neumann algebra $\mathcal{M}$ and a normal, faithful, tracial state τ, the *Fuglede–Kadison determinant* [7] is the map $\Delta = \Delta_\tau : \mathcal{M} \to [0, \infty)$ defined by

$$\Delta(T) = \exp\big(\tau(\log|T|)\big) := \lim_{\epsilon \to 0^+} \exp\big(\tau(\log|T| + \epsilon)\big).$$

Fuglede and Kadison proved that it is multiplicative: $\Delta(AB) = \Delta(A)\Delta(B)$.

The *Brown measure* ν_T was introduced by L.G. Brown [2]. It is a sort of spectral distribution measure for elements $T \in \mathcal{M}$ (and for certain unbounded operators affiliated to $\mathcal{M}$). It is defined to be the Laplacian (in the sense of distributions in $\mathbb{C}$) of the function $f(\lambda) = \frac{1}{2\pi}\log\Delta(T-\lambda)$; Brown proved, among other properties, that it is a probability measure whose support is contained in the spectrum of T. Later, Haagerup and Schultz [9] proved that the Fuglede–Kadison determinant and Brown measure are defined and have nice properties for all closed, densely defined, possibly unbounded operators T affiliated to $\mathcal{M}$ such that $\tau(\log^+|T|) < \infty$, where $\log^+(x) = \max(\log(x), 0)$. We will use the notation $\exp(\mathcal{L}_1)(\mathcal{M}, \tau)$ for this set. It is easy to see that $\exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ is an $\mathcal{M}$-bimodule; it is, in fact, a $*$-algebra containing $\mathcal{M}$ as a $*$-subalgebra (see [6]). A characterization (Theorem 2.7 of [9]) of the Brown measure ν_T of $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ is as the unique probability measure on $\mathbb{C}$ satisfying

$$\int_{\mathbb{C}} \log^+|z|\, d\nu_T(z) < \infty \tag{1}$$

and

$$\int_{\mathbb{C}} \log|z-\lambda|\, d\nu_T(z) = \log\Delta(T-\lambda) \qquad (\lambda \in \mathbb{C}). \tag{2}$$

Brown measure is naturally defined on elements of $\exp(\mathcal{L}^1)$; we will need reduction theory also for unbounded operators in Hilbert space. Nussbaum [11] introduced this theory and developed several aspects of it. In this paper, we will prove and make use of