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An eigensystem approach to Anderson localization

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ABSTRACT

We introduce a new approach for proving localization (pure point spectrum with exponentially decaying eigenfunctions, dynamical localization) for the Anderson model at high disorder. In contrast to the usual strategy, we do not study finite volume Green's functions. Instead, we perform a multiscale analysis based on finite volume eigensystems (eigenvalues and eigenfunctions). Information about eigensystems at a given scale is used to derive information about eigensystems at larger scales. This eigensystem multiscale analysis treats all energies of the finite volume operator at the same time, establishing level spacing and localization of eigenfunctions in a fixed box with high probability. A new feature is the labeling of the eigenvalues and eigenfunctions by the sites of the box.

Contents

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0. Introduction

The Anderson model [8] is the prototype for the study of localization properties of quantum states of single electrons in disordered solids. It is given by a random Schrödinger operator $H_{\varepsilon,\boldsymbol{\omega}} = -\varepsilon\Delta + V_{\boldsymbol{\omega}}$ acting on $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian, $V_{\boldsymbol{\omega}}$ is a random potential, and $\varepsilon > 0$ is the reciprocal of the disorder parameter (see Definition 1.1 for the details). The basic phenomenon, known as the Anderson localization, is that high disorder ($\epsilon \ll 1$) leads to localization of electron states. Its most basic manifestation is that $H_{\varepsilon,\boldsymbol{\omega}}$ has pure point spectrum with exponentially decaying eigenfunctions with probability one: for almost every configuration of the random potential, $H_{\varepsilon,\boldsymbol{\omega}}$ has a complete orthonormal basis of eigenvalues $\{\psi_{\varepsilon,\boldsymbol{\omega},j}\}_{j\in\mathbb{N}}$ such that

$$|\psi_{\varepsilon,\boldsymbol{\omega},j}(x)| \leq C_{\varepsilon,\boldsymbol{\omega},j} \mathrm{e}^{-m_\varepsilon \|x\|} \quad \text{for all} \quad x \in \mathbb{Z}^d \quad \text{and} \quad j \in \mathbb{N},$$

where $m_\varepsilon > 0$, the reciprocal of the localization length, is nonrandom and independent of $j \in \mathbb{N}$. Other manifestations include dynamical localization and SULE (semi-uniformly localized eigenfunctions). (See, for example, [6,38,40].)

These manifestations of localization suggest that truncation of the system to a finite box Λ_L of side $L \gg \frac{1}{m_\varepsilon}$ should not affect localization properties deep inside the box. This leads to the expectation that if one could establish an appropriate analogue of localization for a sequence of boxes Λ_{L_n}, with $L_n \to \infty$, then localization should hold in the whole of $\mathbb{Z}^d$ as well. This strategy can indeed be implemented and is known as the multiscale analysis. In a nutshell, the multiscale analysis uses as input localizing properties at scale L_n to establish localizing properties at scale L_{n+1}. The question is what kind of information we want to carry from scale to scale. In the traditional approach to Anderson localization, such information is encoded in the decay properties of the underlying Green's function. For single-particle systems, the Green's function