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# On the existence of Euler–Lagrange orbits satisfying the conormal boundary conditions

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ABSTRACT

Let $(M, g)$ be a closed connected Riemannian manifold, $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian. Given two non-empty closed submanifolds $Q_0, Q_1 \subseteq M$ and a real number $k$, we study the existence of Euler–Lagrange orbits with energy $k$ connecting $Q_0$ to $Q_1$ and satisfying suitable boundary conditions, known as *conormal boundary conditions*. We introduce the Mañé critical value which is relevant for this problem and discuss existence results for supercritical and subcritical energies. We also provide counterexamples showing that all the results are sharp.


## 1. Introduction

Let $(M, g)$ be a closed connected Riemannian manifold and let $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian (that is a smooth fiberwise $C^2$-strictly convex and superlinear function). The Euler–Lagrange equation, which in local coordinates is given by

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma, \dot{\gamma}) - \frac{\partial L}{\partial q}(\gamma, \dot{\gamma}) = 0,$$

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gives rise to a flow on $TM$, known as the *Euler–Lagrange flow.* The energy function

$$E(q,v) = d_v L(q,v) \cdot v - L(q,v)$$

associated with $L$ is a prime integral of the motion, meaning that it is constant along solutions of the Euler–Lagrange equation. Moreover, $E$ is Tonelli and attains its minimum at $v = 0$; in particular, the energy level sets $E^{-1}(k)$ are compact and invariant under the Euler–Lagrange flow, which therefore turns out to be complete on $TM$. Here we are interested in the following

**Question.** *Given two non-empty closed submanifolds $Q_0, Q_1 \subseteq M$, for which $k \in \mathbb{R}$ does there exist an Euler–Lagrange orbit $\gamma$ with energy $k$ and satisfying the* conormal boundary conditions*?*

Without loss of generality we may suppose $Q_0, Q_1$ connected. Recall that an Euler–Lagrange orbit $\gamma : [0,R] \to M$ is said to satisfy the *conormal boundary conditions* if

$$\begin{cases} \gamma(0) \in Q_0, \ \ \gamma(R) \in Q_1, \\ d_v L(\gamma(0), \dot\gamma(0))\Big|_{T_{\gamma(0)}Q_0} = 0, \\ d_v L(\gamma(R), \dot\gamma(R))\Big|_{T_{\gamma(R)}Q_1} = 0. \end{cases} \tag{1.1}$$

In the case of geodesic flows (i.e. when $L$ is just the kinetic energy defined by $g$), one is simply requiring that $\gamma$ is a geodesic hitting $Q_0$ and $Q_1$ orthogonally. For sake of conciseness, throughout the paper we will call solutions of (1.1) simply *connecting orbits.*

The question above can also be formulated in the Hamiltonian setting. Let $H : T^*M \to \mathbb{R}$ be the Tonelli Hamiltonian given by the Fenchel dual of $L$

$$H(q,p) = \max_{v \in T_qM} \Big[\langle p, v\rangle_q - L(q,v)\Big], \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between tangent and cotangent bundle. For which $k \in \mathbb{R}$ does $H^{-1}(k)$ carry a Hamiltonian orbit $u : [0,R] \to T^*M$ with

$$u(0) \in N^*Q_0, \quad u(R) \in N^*Q_1?$$

Here, for $i = 0,1$, $N^*Q_i$ is the *conormal bundle* of $Q_i$

$$N^*Q_i := \Big\{(q,p) \in T^*M \ \Big|\ q \in Q_i,\ T_qQ_i \subseteq \ker p\Big\}.$$

We refer to [3,15] or [21, Section 6.4] for general facts and properties of conormal bundles.