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The fractional Laplacian in power-weighted L^p spaces: Integration-by-parts formulas and self-adjointness

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A B S T R A C T

We consider the fractional Laplacian operator $(-\Delta)^s$ (let $s \in (0,1)$) on Euclidean space and investigate the validity of the classical integration-by-parts formula that connects the $L^2(\mathbb{R}^d)$ scalar product between a function and its fractional Laplacian to the nonlocal norm of the fractional Sobolev space $\dot{H}^s(\mathbb{R}^d)$. More precisely, we focus on functions belonging to some weighted L^2 space whose fractional Laplacian belongs to another weighted L^2 space: we prove and disprove the validity of the integration-by-parts formula depending on the behaviour of the weight $\rho(x)$ at infinity. The latter is assumed to be like a power both near the origin and at infinity (the two powers being possibly different). Our results have direct consequences for the self-adjointness of the linear operator formally given by $\rho^{-1}(-\Delta)^s$. The generality of the techniques developed allows us to deal with weighted L^p spaces as well.

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1. Introduction

Given $d \in \mathbb{N}$ and any $s \in (0,1)$, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^d$ is a nonlocal operator defined on test functions by

$$(-\Delta)^s(\phi)(x) := C_{d,s}\ p.v. \int_{\mathbb{R}^d} \frac{\phi(x)-\phi(y)}{|x-y|^{d+2s}}\,\mathrm{d}y \quad \forall x \in \mathbb{R}^d\,, \ \ \forall \phi \in \mathcal{D}(\mathbb{R}^d)\,,$$

where *p.v.* denotes the *principal value* of the integral about x and $C_{d,s}$ is a suitable positive constant depending only on d and s, such that $\lim_{s\to 1^-}(-\Delta)^s(\phi) = -\Delta\phi$ (see for instance [15, Sections 3, 4]). An alternative representation of $(-\Delta)^s$ is the one involving the celebrated extension of Caffarelli and Silvestre [10], where the fractional Laplacian of ϕ is seen as the trace of the normal derivative of the harmonic extension of ϕ in the upper half-plane (at least for $s = \frac{1}{2}$, while for a general $s \in (0,1)$ one has to introduce a suitable degenerate or singular elliptic operator). Even though it has proved to be a very powerful tool in dealing with issues related to the fractional Laplacian, we shall no further consider the aforementioned extension, since our arguments need not take advantage of it.

A Sobolev space naturally associated with the fractional Laplacian is $\dot{\mathrm{H}}^s(\mathbb{R}^d)$, namely the closure of $\mathcal{D}(\mathbb{R}^d)$ endowed with the norm

$$\|\phi\|_{\dot{\mathrm{H}}^s(\mathbb{R}^d)} := \left\|(-\Delta)^{s/2}(\phi)\right\|_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)\,.$$

A well-known result (see [15, Proposition 3.6]) asserts that

$$\|\phi\|^2_{\dot{\mathrm{H}}^s(\mathbb{R}^d)} = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)\,, \tag{1.1}$$

so that we can equivalently define $\dot{\mathrm{H}}^s(\mathbb{R}^d)$ by means of the nonlocal (squared) norm appearing in the r.h.s. of (1.1). Let us point out that by $\mathrm{H}^s(\mathbb{R}^d)$ one usually means the space of functions $v \in \mathrm{L}^2(\mathbb{R}^d)$ such that $\|v\|_{\dot{\mathrm{H}}^s(\mathbb{R}^d)} < \infty$, which in fact coincides with $\mathrm{L}^2(\mathbb{R}^d) \cap \dot{\mathrm{H}}^s(\mathbb{R}^d)$. However, since below we shall deal with functions belonging to some *weighted* L^2 spaces (L^p in general), throughout the paper we shall never make use of $\mathrm{H}^s(\mathbb{R}^d)$.

By means of classical Fourier-transform arguments (we refer again to [15, Section 3]), it is straightforward to show that if $v \in \mathrm{L}^2(\mathbb{R}^d)$ and $(-\Delta)^s(v) \in \mathrm{L}^2(\mathbb{R}^d)$ (to be understood in the distributional sense), then $v \in \mathrm{H}^s(\mathbb{R}^d)$. Moreover, since $\|\cdot\|_{\dot{\mathrm{H}}^s(\mathbb{R}^d)}$ naturally induces an inner product $\langle\cdot,\cdot\rangle_{\dot{\mathrm{H}}^s(\mathbb{R}^d)}$, the following *integration-by-parts* formulas hold: