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Maximal decay inequalities for trilinear oscillatory integrals of convolution type

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ABSTRACT

In this paper we prove sharp L^∞–L^∞–L^∞ decay for certain trilinear oscillatory integral forms of convolution type on $\mathbb{R}^2$. These estimates imply earlier L^2–L^2–L^2 results obtained by the second author as well as corresponding sharp, stable sublevel set estimates of the form studied by Christ [3] and Christ, Li, Tao, and Thiele [5]. New connections to the multilinear results of Phong, Stein, and Sturm [13] are also considered.

1. Introduction

Beginning with the groundbreaking work of Christ, Li, Tao, and Thiele [5], there has been significant interest in the harmonic analysis literature to develop a robust and general theory of multilinear oscillatory integral operators. However, despite the fundamental insights provided by [5], progress on this program has been slow, due to the sheer complexity of the problem and the apparent inadequacy of existing tools in this

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more general context. Some of the more successful strategies to date focus on special cases of their general framework; see [3,4,6,8,9,15] for recent progress and [1,13,2] for other related topics. One such special case, which we will further examine in this paper, is the case of trilinear forms which have convolution-type structure. Specifically we will consider forms $\Lambda(f,g,h)$ given by

$$\Lambda(f,g,h) = \iint e^{i\lambda S(x,y)} f(x)g(y)h(x+y)\phi(x,y)dxdy, \tag{1.1}$$

where $S(x,y)$ is a real analytic function defined in a neighborhood of the origin and ϕ is smooth cut-off function supported sufficiently near the origin. In the present case, we will take f,g,h to belong to $L^p(\mathbb{R})$, $L^q(\mathbb{R})$, and $L^r(\mathbb{R})$, respectively, and study the norm of the form as a multilinear functional on $L^p \times L^q \times L^r$ as $\lambda \to \pm\infty$. As observed in [5], one expects no decay at all when $S(x,y)$ may be written as a sum $S_1(x)+S_2(y)+S_3(x+y)$ for measurable functions S_1, S_2, and S_3. When S is smooth, the (non-)degeneracy of S is captured by the action of the differential operator D

$$D = \partial_x\partial_y(\partial_x - \partial_y)$$

which annihilates sums of the form $S_1(x)+S_2(y)+S_3(x+y)$.

In his thesis, the second author considered a special case, namely f, g and h all belonging to L^2, and showed that sharp decay estimates can be characterized by the relative multiplicity of S, which is defined to be the multiplicity of the quotient of S by the class of functions annihilated by the differential operator D. More precisely, if $n \in \mathbb{N}$ denotes the relative multiplicity of S, then

$$|\Lambda(f,g,h)| \lesssim |\lambda|^{-\frac{1}{2n}} \|f\|_2 \|g\|_2 \|h\|_2. \tag{1.2}$$

The basic observation upon which these earlier results is based is an estimate of the form

$$|\Lambda(f,g,h)| \leq C|\lambda|^{-\frac{1}{6}} \|f\|_2 \|g\|_2 \|h\|_2 \tag{1.3}$$

for phases S with $|DS| > c > 0$ on the support of ϕ, which itself follows from a clever application (see [11]) of the Hörmander argument for non-degenerate bilinear oscillatory integral forms. Despite the fact that scaling arguments show that the exponent of λ in (1.3) cannot be improved, a comparison to the sublevel set estimate (namely, the decay rate of the volume of the set

$$\left\{(x,y) \in [0,1]^2 : |S(x,y)| < \epsilon\right\}$$

as $\epsilon \to 0^+$) suggests that the decay rate $|\lambda|^{-1/6}$ is likely not the best possible if one considers L^p spaces on the right-hand side other than L^2. Our first result is that such an improvement over $\lambda^{-1/6}$ can indeed be achieved: