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Gap eigenvalues and asymptotic dynamics of geometric wave equations on hyperbolic space [☆]

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In this paper we study k-equivariant wave maps from the hyperbolic plane into the 2-sphere as well as the energy critical equivariant $SU(2)$ Yang–Mills problem on 4-dimensional hyperbolic space. The latter problem bears many similarities to a 2-equivariant wave map into a surface of revolution. As in the case of 1-equivariant wave maps considered in [9], both problems admit a family of stationary solutions indexed by a parameter that determines how far the image of the map wraps around the target manifold. Here we show that if the image of a stationary solution is contained in a geodesically convex subset of the target, then it is asymptotically stable in the energy space. However, for a stationary solution that covers a large enough portion of the target, we prove that the Schrödinger operator obtained by linearizing about such a harmonic map admits a simple positive eigenvalue in the spectral gap. As there is no a priori nonlinear obstruction to asymptotic stability, this gives evidence for the existence of

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metastable states (i.e., solutions with anomalously slow decay rates) in these simple geometric models.

1. Introduction

We consider the k-equivariant wave maps equation from $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{S}^2$ and the energy critical equivariant Yang–Mills problem on $\mathbb{R} \times \mathbb{H}^4$ with gauge group $SU(2)$. After the usual equivariant reductions, both equations take the form

$$\psi_{tt} - \psi_{rr} - \coth r\, \psi_r + k^2 \frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0, \tag{1.1}$$

where (ψ, θ) are geodesic polar coordinates on a target surface of revolution $\mathcal{M}$, and g determines the metric, $ds^2 = d\psi^2 + g^2(\psi)d\theta^2$. In the case of k-equivariant wave maps, we set $\mathcal{M} = \mathbb{S}^2$ and $g(\psi) = \sin\psi$. For the Yang–Mills problem, $k = 2$ and $g(\psi) = \psi - \frac{1}{2}\psi^2$. Note that one can view the latter problem as a 2-equivariant wave map from $\mathbb{R} \times \mathbb{H}^2$ into a surface of revolution which is diffeomorphic to $\mathbb{S}^2$ by restricting to $\psi \in [0, 2]$. Indeed, $g(0) = g(2) = 0$, $\psi = 1$ is the unique zero of $g'(\psi)$ and $\{\psi \leq 1\}$ defines the largest geodesically convex neighborhood of the "north pole", $\psi = 0$.

There are several features of these models that make them interesting testing grounds in the study of asymptotic dynamics of dispersive equations on curved spaces. As we will see below, the introduction of hyperbolic geometry on the domain allows for an abundance of finite energy stationary solutions to (1.1) – these are harmonic maps in the case of the wave maps equation. The curved background also eliminates any natural scaling invariance for the problem, thus removing an a priori obstruction to the asymptotic stability of stationary solutions. However the energy-criticality of these equations is still manifest since solutions that concentrate at very small scales can be well approximated by solutions to the underlying Euclidean problems, which are energy critical; to obtain the scale invariant Euclidean equation simply replace $\coth r$ by r^{-1} and $\sinh^2 r$ by r^2 in (1.1). Taking this last point a bit further, one expects that the fundamental blow-up constructions for the corresponding Euclidean models of Krieger, Schlag, Tataru [7,8], Rodnianski, Sterbenz [12], and Raphaël, Rodnianski [11] should carry over to the present hyperbolic setting.

In this paper, we study a more subtle effect of the underlying scale invariant Euclidean equation on the asymptotic dynamics of the hyperbolic problem (1.1), namely the formation of eigenvalues in the spectral gap of the Schrödinger operator obtained by linearizing about a stationary solution to (1.1) that wraps sufficiently far around the target manifold $\mathcal{M}$. This phenomenon was discovered in [9] in the case of 1-equivariant wave maps from $\mathbb{R} \times \mathbb{H}^2$ to $\mathbb{S}^2$. Here we establish the existence of gap eigenvalues for higher equivalence classes $k \geq 2$, and for the equivariant Yang–Mills problem, while fur-