Journal of Functional Analysis 271 (2016) 3186–3241

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Bochner–Riesz profile of anharmonic oscillator $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$

Peng Chen [a], Waldemar Hebisch [b], Adam Sikora [c,*]

[a] School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, PR China
[b] Mathematical Institute, University of Wroclaw, Pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
[c] Department of Mathematics, Macquarie University, NSW 2109, Australia

ARTICLE INFO

Article history:
Received 15 June 2015
Accepted 24 August 2016
Available online 31 August 2016
Communicated by L. Saloff-Coste

MSC:
42B15
42B20
47F05

Keywords:
Spectral multipliers
Schrödinger operators
Bochner–Riesz means
Airy function

ABSTRACT

We investigate spectral multipliers, Bochner–Riesz means and the convergence of eigenfunction expansion corresponding to the Schrödinger operator with anharmonic potential $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$. We show that the Bochner–Riesz profile of the operator $\mathcal{L}$ completely coincides with such profile of the harmonic oscillator $\mathcal{H} = -\frac{d^2}{dx^2} + x^2$. It is especially surprising because the Bochner–Riesz profile of the one-dimensional standard Laplace operator is known to be essentially different and the case of operators $\mathcal{H}$ and $\mathcal{L}$ resembles more the profile of multidimensional Laplace operators. Another surprising element of the main obtained result is the fact that the proof is not based on restriction type estimates and instead an entirely new perspective has to be developed to obtain the critical exponent for Bochner–Riesz means convergence.

Contents

1. Introduction 3187

* Corresponding author.
E-mail addresses: chenpeng3@mail.sysu.edu.cn (P. Chen), hebisch@math.uni.wroc.pl (W. Hebisch), adam.sikora@mq.edu.au (A. Sikora).

http://dx.doi.org/10.1016/j.jfa.2016.08.016

2. Eigenfunction expansion of the operator $\mathcal{L}$ 3191
3. Spectral multiplier theorems for abstract self-adjoint operators 3194
4. More light on Airy function 3202
5. Proof of Theorem 1.2 3214
6. Necessary conditions for Bochner–Riesz summability and proof of Theorem 1.3 3234
7. Concluding remarks 3238
Acknowledgments 3240
References 3240

1. Introduction

One of the most significant and central problems in harmonic analysis is convergence of the Fourier transform and series. This problem leads in a natural way to the question of convergence of Bochner–Riesz means of Fourier integrals and series. In a systematic manner the topic was initiated in the 1930s by Bochner. Since then it has attracted very significant attention. Nevertheless there still remain many fundamental problems to be resolved. Detailed account of the main ideas and development of this area can be found for example in [15, Chapter 8], [37, Section IX.2], [35, Chapter II], [40] or [25].

Using the language of the spectral theory the problem of Convergence of Bochner–Riesz means of Fourier series can be formulated for any eigenfunction expansion of any abstract self-adjoint operators. Convergence and equivalently boundedness of Bochner–Riesz means for general differential operators or various specific operators were studied among others by Christ, Karadzhov, Koch, Ricci, Seeger, Sogge, Stempak, Tataru, Thangavelu and Zienkiewicz, see [8,22–24,31,34,36,38,42]. See also [16]. This paper is a continuation of these efforts in particular case of the operator $\mathcal{L} = -\frac{d^2}{dx^2} + |x|$.

The theory of L^p spectral multipliers is essentially equivalent to Bochner–Riesz analysis but is more flexible and precise, see discussion in Section 7. Therefore we adopt this approach in this paper and we state our main result Theorem 1.2 below in the language of spectral multipliers. In this context it is worth mentioning that the theory of L^p spectral multipliers itself also attracts significant interest. Initially spectral theory for self-adjoint operators was motivated by Fourier multiplier type results of Mikhlin and Hörmander [19,28]. These results restricted to radial Fourier multipliers can be written in terms of spectral multipliers for the standard Laplace operator and opened question of possible generalisation to larger class of self-adjoint operators, see also discussion in [7]. In our approach we investigate Mikhlin and Hörmander multipliers together with Bochner–Riesz analysis as essentially the same research area. The literature devoted to the spectral multipliers is much too broad to be listed here so we refer the reader to [10,5,11,16] for large class of examples of papers devoted to this area of harmonic analysis. Some recent developments going in somehow different direction can be found in [26]. A few other interesting examples of spectral multiplier results in various settings can be found in [1,2,8,9,27,29,31,35,34,41].

In [42] Thangavelu showed that the profiles of Bochner–Riesz means convergence for the standard Laplace operator in one dimension and one dimensional harmonic oscillator