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Interpolation inequalities in pattern formation

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ABSTRACT

We prove some interpolation inequalities which arise in the analysis of pattern formation in physics. They are the strong version of some already known estimates in weak form that are used to give a lower bound of the energy in many contexts (coarsening and branching in micromagnetics and superconductors). The main ingredient in the proof of our inequalities is a geometric construction which was first used by Choksi, Conti, Kohn, and one of the authors in [4] in the study of branching in superconductors.

1. Introduction

In this paper we establish some interpolation inequalities which are connected with the study of certain physical phenomena. More precisely, the inequalities that we prove are the strong versions of some already known interpolation estimates in weak form that play a crucial role in the study of pattern formation in physics.

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In many physical phenomena described by a variational model, in order to understand why certain patterns are observed, it is natural to study the features of pattern with energy close to the minimal one. This requires a good understanding of at least the scaling of the minimal energy in terms of the model parameters. Upper bounds on the minimal energy are obtained by physically motivated trial patterns (Ansatz). Typically, Ansatz-free lower bounds rely on suitable interpolation inequalities that involve some functional norms related to the energy under consideration.

The first two interpolation inequalities that we present here involve the BV-norm and the $\dot{H}^{-1}$-norm of a function u. The first estimate holds in any dimension d and it is stated in Theorem 1.1 below. The second inequality (see Theorem 1.2) holds in dimension 2 for functions bounded below, and it improves the result in Theorem 1.1 by a logarithmic factor.

Here with $\dot{H}^{-1}$ we refer to the homogeneous negative Sobolev space H^{-1}. Let u be a periodic function defined on the torus $[0,\Lambda]^d$ and with vanishing average. We recall that the $\dot{H}^{-1}$-norm of u is defined as follows

$$\|u\|_{\dot{H}^{-1}} = \||\nabla|^{-1}u\|_2^2 := \inf_j \left\{ \int_{[0,\Lambda]^d} |j|^2 \;\middle|\; j \text{ is periodic and } \nabla\cdot j = u \right\}$$
$$= \int_{[0,\Lambda]^d} |\nabla\varphi|^2 \quad \text{where } -\Delta\varphi = u.$$

It can also be defined via Fourier transform:

$$\||\nabla|^{-1}u\|_2^2 = \int (|k|^{-1}|F(u)|)^2 dk,$$

where $\int dk$ has to be interpreted in a discrete sense.

We recall moreover that the BV-seminorm of a function u in $L^1([0,\Lambda]^d)$, is given by

$$|u|_{BV} := \sup\left\{ \int_{[0,\Lambda]^d} u\,\mathrm{div}\phi \;\text{ with }\; \phi\in C_c^1([0,\Lambda]^d,\mathbb{R}^n),\; |\phi|\le 1 \right\}, \tag{1.1}$$

were $C_c^1([0,\Lambda]^d;\mathbb{R}^n)$ denotes the class of C^1 vector fields compactly supported in $[0,\Lambda]^d$.

Throughout the paper, we will denote the BV-seminorm of a function u using indifferently the notations $\|\nabla u\|_1$ and $\int|\nabla u|$. Hence, in all our statements and in all the computations, the symbols $\|\nabla u\|_1$ and $\int|\nabla u|$ refer to the total variation of u, rather than to the L^1-norm of its weak derivatives.

Theorem 1.1. *Let $u:[0,\Lambda]^d\to\mathbb{R}$ be a periodic function such that $\int_{[0,\Lambda]^d} u = 0$.*