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## Noncommutative reproducing kernel Hilbert spaces



Joseph A. Ball<sup>a,\*</sup>, Gregory Marx<sup>a</sup>, Victor Vinnikov<sup>b</sup>

<sup>a</sup> Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA

<sup>b</sup> Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

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### ABSTRACT

The theory of positive kernels and associated reproducing kernel Hilbert spaces, especially in the setting of holomorphic functions, has been an important tool for the last several decades in a number of areas of complex analysis and operator theory. An interesting generalization of holomorphic functions, namely free noncommutative functions (e.g., functions of square-matrix arguments of arbitrary size satisfying additional natural compatibility conditions), is now an active area of research, with motivation and applications from a variety of areas (e.g., noncommutative functional calculus, free probability, and optimization theory in linear systems engineering). The purpose of this article is to develop a theory of positive kernels and associated reproducing kernel Hilbert spaces for the setting of free noncommutative function theory.

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\* Corresponding author.

E-mail addresses: [joball@math.vt.edu](mailto:joball@math.vt.edu) (J.A. Ball), [marxg@vt.edu](mailto:marxg@vt.edu) (G. Marx), [vinnikov@cs.bgu.ac.il](mailto:vinnikov@cs.bgu.ac.il) (V. Vinnikov).

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## 1. Introduction

The goal of the present paper is to incorporate the classical theory of positive kernels and reproducing kernel Hilbert spaces (see [4,1]) into the new setting of free noncommutative function theory (see [29]).

We use the following operator-valued adaptation of the notion of positive kernel developed in some depth by Aronszajn in [4]. Let  $\Omega$  be a point set and  $K$  a function from the Cartesian product set  $\Omega \times \Omega$  into the space  $\mathcal{L}(\mathcal{Y})$  of bounded linear operators on a Hilbert spaces  $\mathcal{Y}$ . We say that  $K$  is a **positive kernel** if

$$\sum_{i,j=1}^N \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{E}} \geq 0 \tag{1.1}$$

for all  $\omega_1, \dots, \omega_N \in \Omega$ ,  $y_1, \dots, y_N \in \mathcal{Y}$ ,  $N = 1, 2, \dots$ . Equivalent conditions are:

- There is a Hilbert space  $\mathcal{H}(K)$  consisting of  $\mathcal{Y}$ -valued functions on  $\Omega$  such that  $K$  has the following **reproducing kernel** property with respect to  $\mathcal{H}(K)$ :
  - (1) for any  $\omega \in \Omega$  and  $y \in \mathcal{Y}$  the function  $K_{\omega,y}$  given by  $K_{\omega,y}(\omega') = K(\omega', \omega)y$  belongs to  $\mathcal{H}(K)$ , and
  - (2) for all  $f \in \mathcal{H}(K)$  and  $y \in \mathcal{Y}$ , the reproducing property

$$\langle f, K_{\omega,y} \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}} \tag{1.2}$$

holds.

- There is a Hilbert space  $\mathcal{X}$  and a function  $H: \Omega \rightarrow \mathcal{L}(\mathcal{H}(K), \mathcal{Y})$  so that the following **Kolmogorov decomposition** holds:

$$K(\omega', \omega) = H(\omega')H(\omega)^*. \tag{1.3}$$

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