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# Noncommutative reproducing kernel Hilbert spaces

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A B S T R A C T

The theory of positive kernels and associated reproducing kernel Hilbert spaces, especially in the setting of holomorphic functions, has been an important tool for the last several decades in a number of areas of complex analysis and operator theory. An interesting generalization of holomorphic functions, namely free noncommutative functions (e.g., functions of square-matrix arguments of arbitrary size satisfying additional natural compatibility conditions), is now an active area of research, with motivation and applications from a variety of areas (e.g., noncommutative functional calculus, free probability, and optimization theory in linear systems engineering). The purpose of this article is to develop a theory of positive kernels and associated reproducing kernel Hilbert spaces for the setting of free noncommutative function theory.


**Contents**



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## 1. Introduction

The goal of the present paper is to incorporate the classical theory of positive kernels and reproducing kernel Hilbert spaces (see [4,1]) into the new setting of free noncommutative function theory (see [29]).

We use the following operator-valued adaptation of the notion of positive kernel developed in some depth by Aronszajn in [4]. Let $\Omega$ be a point set and $K$ a function from the Cartesian product set $\Omega \times \Omega$ into the space $\mathcal{L}(\mathcal{Y})$ of bounded linear operators on a Hilbert spaces $\mathcal{Y}$. We say that $K$ is a **positive kernel** if

$$\sum_{i,j=1}^{N} \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{E}} \geq 0 \tag{1.1}$$

for all $\omega_1, \dots, \omega_N \in \Omega$, $y_1, \dots, y_N \in \mathcal{Y}$, $N = 1, 2, \dots$ . Equivalent conditions are:

- There is a Hilbert space $\mathcal{H}(K)$ consisting of $\mathcal{Y}$-valued functions on $\Omega$ such that $K$ has the following **reproducing kernel** property with respect to $\mathcal{H}(K)$:
  (1) for any $\omega \in \Omega$ and $y \in \mathcal{Y}$ the function $K_{\omega,y}$ given by $K_{\omega,y}(\omega') = K(\omega', \omega)y$ belongs to $\mathcal{H}(K)$, and
  (2) for all $f \in \mathcal{H}(K)$ and $y \in \mathcal{Y}$, the reproducing property
  $$\langle f,\, K_{\omega,y} \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}} \tag{1.2}$$
  holds.
- There is a Hilbert space $\mathcal{X}$ and a function $H \colon \Omega \to \mathcal{L}(\mathcal{H}(K), \mathcal{Y})$ so that the following **Kolmogorov decomposition** holds:
  $$K(\omega', \omega) = H(\omega') H(\omega)^*. \tag{1.3}$$