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High-power asymptotics of some weighted harmonic Bergman kernels

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For weights ρ which are either radial on the unit ball or depend only on the vertical coordinate on the upper half-space, we describe the asymptotic behaviour of the corresponding weighted harmonic Bergman kernels with respect to ρ^α as $\alpha \to +\infty$. This can be compared to the analogous situation for the holomorphic case, which is of importance in the Berezin quantization as well as in complex geometry.

1. Introduction

Let Ω be a domain in $\mathbf{C}^n$, ρ a positive smooth ($= C^\infty$) weight on Ω, $L^2_{\text{hol}}(\Omega, \rho^\alpha)$ the subspace of all holomorphic functions in the weighted Lebesgue space $L^2(\Omega, \rho^\alpha)$, and $K_\alpha(x, y)$ the reproducing kernel for $L^2_{\text{hol}}(\Omega, \rho^\alpha)$, i.e. the weighted Bergman kernel on Ω with respect to the weight ρ^α. Under suitable hypothesis on Ω and ρ (namely, for Ω

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bounded and pseudoconvex, $\log\frac{1}{\rho}$ strictly plurisubharmonic, and ρ a defining function for Ω, i.e. vanishing to precisely the first order at the boundary), it is then known that

$$K_\alpha(x,x) \sim \frac{\alpha^n}{\pi^n \rho(x)^\alpha} \det\left[\partial\overline{\partial}\log\frac{1}{\rho(x)}\right] \qquad \text{as } \alpha \nearrow +\infty. \tag{1}$$

In fact, there is even a similar result for $K_\alpha(x,y)$ with y close to x, and one also has a complete asymptotic expansion as $\alpha \nearrow +\infty$

$$K_\alpha(x,y) \approx \frac{\alpha^n}{\pi^n \rho(x,y)^\alpha} \sum_{j=0}^{\infty} \frac{b_j(x,y)}{\alpha^j}, \qquad b_0(x,x) = \det[\partial\overline{\partial}\log\tfrac{1}{\rho(x)}], \tag{2}$$

with some "sesqui-analytic extension" $\rho(x,y)$ of $\rho(x)$ and sesqui-analytic coefficient functions $b_j(x,y)$. Furthermore one can differentiate (1) and (2) termwise any number of times. There are, finally, variants also for the weighted Bergman spaces with respect to $\rho^\alpha\psi^m$, where ψ is another weight function satisfying the same hypotheses as ρ and $m \geq 0$ is a fixed integer. All these "high power asymptotics" can also be extended from functions on domains Ω to sections of holomorphic Hermitian line bundles over a manifold Ω, and are then of central importance in certain approaches to quantization (the Berezin–Toeplitz quantization procedure), as well as in complex geometry (where (1) is sometimes known as the Tian–Yau–Zelditch expansion, and plays prominent role e.g. in connection with semistability and constant scalar curvature metrics on Ω); see for instance Berezin [2], Engliš [7,9], Zelditch [16], Catlin [4], Donaldson [6], and the references therein.

While there exist several well-understood variants of methods how to prove (1) (or (2)) nowadays, none of them makes it quite clear what does the holomorphy of functions in L^2_{hol} have to do with (1), (2) or with the coefficients b_j above; in fact, a priori there is little reason to expect that holomorphic functions should have anything to do either with quantization or with constant scalar curvature metrics, and one is just left to wonder at Berezin's original insight in noticing (1) and its applications. In particular, it remains quite elusive what happens for other reproducing kernel subspaces in $L^2(\Omega,\rho^\alpha)$.

The goal of this paper is to explore the analogue of (1) for the spaces of harmonic, rather than holomorphic, functions, i.e. for the reproducing kernels $R_\alpha(x,y)$ — the harmonic Bergman kernels — of the subspaces $L^2_{\text{harm}}(\Omega,\rho^\alpha)$ of all harmonic functions in $L^2(\Omega,\rho^\alpha)$.

In the holomorphic setting, the simplest examples for (1) and (2) are the standard weighted Bergman spaces on the unit disc $\mathbf{D}$ in $\mathbf{C}$ with $\rho(z) = 1-|z|^2$, when

$$K_\alpha(x,y) = \tfrac{\alpha+1}{\pi}(1-x\overline{y})^{-\alpha-2}; \tag{3}$$

or, equivalently (via the Cayley transform), on the upper half-plane $\{z : \operatorname{Im} z > 0\}$ in $\mathbf{C}$ with $\rho(z) = \operatorname{Im} z$ and