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Distances between transition probabilities of diffusions and applications to nonlinear Fokker–Planck–Kolmogorov equations

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A B S T R A C T

We estimate the total variation and Kantorovich distances between transition probabilities of two diffusions with different diffusion matrices and drifts via a natural quadratic distance between the drifts and diffusion matrices. Applications to nonlinear Fokker–Planck–Kolmogorov equations, optimal control and mean field games are given.

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1. Introduction

The goal of this paper is to give upper bounds for the total variation, entropy and Kantorovich distances between two probability solutions $\varrho_1(x,t)$ and $\varrho_2(x,t)$ to Fokker–Planck–Kolmogorov equations

$$\partial_t \varrho_k(x,t) = \partial_{x_i}\partial_{x_j}(a_k^{ij}(x,t)\varrho_k(x,t)) - \partial_{x_i}(b_k^i(x,t)\varrho_k(x,t)), \quad k=1,2, \tag{1.1}$$

with different diffusion matrices and drifts on $\mathbb{R}^d \times [0,T]$ with fixed $T>0$. In case of equal initial distributions and identity diffusion matrices, for the entropy of ϱ_2 with respect to ϱ_1 we obtain the estimate

$$\int\limits_{\mathbb{R}^d} \log \frac{\varrho_2(x,t)}{\varrho_1(x,t)}\, \varrho_2(x,t)\,dx \le \frac{1}{2}\int\limits_{\mathbb{R}^d} |b_1(x,t)-b_2(x,t)|^2\,\varrho_2(x,t)\,dx,$$

and for the total variation norm we obtain the estimate

$$\|\varrho_1(\,\cdot\,,t)-\varrho_2(\,\cdot\,,t)\|_{TV}^2 \le \int\limits_0^t \int\limits_{\mathbb{R}^d} |b_1(x,s)-b_2(x,s)|^2\,\varrho_2(x,s)\,dx\,ds.$$

In the general case we obtain quite comparable estimates under rather broad assumptions about our coefficients: the diffusion matrices are locally uniformly elliptic and locally Lipschitzian in space, the drifts are locally bounded, and either some mild integrability conditions are imposed or a certain Lyapunov function exists (an advantage of the latter condition is that it is expressed entirely in terms of the coefficients). In examples we give a number of effectively verified conditions. The principal novelty concerns the case of different diffusion matrices (see comments in Remark 1.5), but also the simpler case of the same diffusion matrix seems to be new. The main result is applied to nonlinear Fokker–Planck–Kolmogorov equations, optimal control and mean field games. Let us explain precisely our framework.

Let us consider a time-dependent second order elliptic operator

$$L_{A,b}u = \sum_{i,j=1}^d a^{ij}\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^d b^i\partial_{x_i}u,$$

where $A(x,t) = (a^{ij}(x,t))_{i,j\le d}$ is a positive symmetric matrix (called the diffusion matrix) with Borel measurable entries and $b(x,t) = (b^i(x,t))_{i=1}^d\colon\ \mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ is a Borel measurable mapping (called the drift coefficient). Suppose that b is locally bounded, i.e., for every ball $U\subset\mathbb{R}^d$, there is a number $B=B(U)\ge 0$ such that

$$|b(x,t)| \le B(U) \quad \forall\, x\in U,\ t\in[0,T],$$

and A is locally Lipschitzian in x and locally strictly positive, i.e.,