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Similarity degree of Fourier algebras ☆

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ABSTRACT

We show that for a locally compact group G, amongst a class which contains amenable and small invariant neighbourhood groups, its Fourier algebra $\mathrm{A}(G)$ satisfies a completely bounded version Pisier's similarity property with similarity degree at most 2. Specifically, any completely bounded homomorphism $\pi : \mathrm{A}(G) \to \mathcal{B}(\mathcal{H})$ admits an invertible S in $\mathcal{B}(\mathcal{H})$ for which $\|S\|\|S^{-1}\| \leq \|\pi\|_{\mathrm{cb}}^2$ and $S^{-1}\pi(\cdot)S$ extends to a $*$-representation of the C*-algebra $\mathcal{C}_0(G)$. This significantly improves some results due to Brannan and Samei (2010) [5] and Brannan, Daws and Samei (2013) [4]. We also note that $\mathrm{A}(G)$ has completely bounded similarity degree 1 if and only if it is completely isomorphic to an operator algebra if and only if G is finite.

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In [32], Pisier launched a major assault on a large class of similarity problems including Kadison's similarity problem for C*-algebras and Dixmier's similarity problem for groups. Pisier's partial solution to Dixmier's problem (see [32, Theorem 3.2] for discrete groups, which is adapted in the third named author's thesis [34] to general locally compact groups) is encapsulated in the following result.

Theorem 0.1. *Let G be a locally compact group. Any bounded homomorphism of the group algebra to the algebra of bounded operators on a Hilbert space, $\pi : \mathrm{L}^1(G) \to \mathcal{B}(\mathcal{H})$, admits an invertible S in $\mathcal{B}(\mathcal{H})$ for which*

(a) $\pi_S = S\pi(\cdot)S^{-1}$ *is a $*$-representation:* $\pi_S(f^*) = \pi_S(f)^*$, *and with*

(b) $\|S\|\|S^{-1}\| \leq K\,\|\pi\|^2$, *for some constant K*

if and only if G is amenable.

We shall refer to success of (a) as the *similarity property* for the algebra $\mathrm{L}^1(G)$. The exponent 2 in (b) means that $\mathrm{L}^1(G)$ has *similarity degree* at most 2.

Our goal is to gain an analogue of this similarity theorem for the Fourier algebra $\mathrm{A}(G)$. This algebra was defined by Eymard [13], and is the Pontryagin dual object to $\mathrm{L}^1(G)$. We note that $\mathrm{A}(G)$ is a Banach algebra of functions on G, which is conjugation-closed and admits spectrum G. In particular, $\mathrm{A}(G)$ is a dense subalgebra of $\mathcal{C}_0(G)$. We succeed in our goal, though only under extra hypotheses on the locally compact group G, as specified in Theorem 2.5, below.

One of the most intriguing aspects of Pisier's methods, is that they add to the convincing body of work which indicates the value of considering the operator space structure to attack certain questions which appear to have no a priori need for such. However, as we explain below, we shall require a natural operator space structure on the Fourier algebra, to frame our problem.

We observe that for abelian G, the Fourier transform of the involution on $\mathrm{L}^1(\widehat{G})$, $f \mapsto f^*$, is given on $\mathrm{A}(G)$ by complex conjugation, $u \mapsto \bar{u}$. Thus this is the suitable involution on $\mathrm{A}(G)$ which we shall consider for any locally compact G. Our goal is to characterize representations $\pi : \mathrm{A}(G) \to \mathcal{B}(\mathcal{H})$ which admit S in $\mathcal{B}(\mathcal{H})$ for which $\pi_S = S\pi(\cdot)S^{-1}$ is a $*$-representation: $\pi_S(\bar{u}) = \pi_S(u)^*$; this is the *similarity property* for $\mathrm{A}(G)$. Because any such homomorphism π must necessarily factor through the minimal operator space of functions $\mathcal{C}_0(G)$ (which is the enveloping C*-algebra of the involutive algebra $\mathrm{A}(G)$ – see beginning of §2), π must be completely bounded. Hence we wish to know if all completely bounded homomorphisms of $\mathrm{A}(G)$ enjoy the similarity property. This was the goal of the investigation of Brannan and the second named author [5] and was followed upon by the same two authors in conjunction with M. Daws [4]. They only gained a complete solution for small invariant neighbourhood (SIN) groups in the first paper, and for amenable groups containing open SIN subgroups in the second. We gain a complete solution for a class of groups containing both amenable and SIN groups (Theorem 2.5). In all their calculations, it was shown that the completely bounded similarity degree is at most 4, i.e. $\|S\|\|S^{-1}\| \leq \|\pi\|_{\mathrm{cb}}^4$ in Theorem 0.1 (b), above,