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A bound for the perimeter of inner parallel bodies

Simon Larson

Department of Mathematics, Royal Institute of Technology, SE-10044 Stockholm, Sweden

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ABSTRACT

We provide a sharp lower bound for the perimeter of the inner parallel sets of a convex body Ω. The bound depends only on the perimeter and inradius r of the original body and states that

$$|\partial\Omega_t| \geq \left(1 - \frac{t}{r}\right)_+^{n-1} |\partial\Omega|.$$

In particular the bound is independent of any regularity properties of $\partial\Omega$. As a by-product of the proof we establish precise conditions for equality. The proof, which is straightforward, is based on the construction of an extremal set for a certain optimization problem and the use of basic properties of mixed volumes.

1. Introduction

Given a convex domain $\Omega \subset \mathbb{R}^n$ we consider the family of its *inner parallel sets*. We denote by Ω_t the inner parallel set at distance $t \geq 0$, which is defined by

$$\Omega_t = \{x \in \Omega : \mathrm{dist}(x, \Omega^c) \geq t\} = \Omega \sim tB.$$

E-mail address: simla@math.kth.se.

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Here B is the unit ball in $\mathbb{R}^n$ and $\sim$ denotes the *Minkowski difference*; a precise definition is given in Section 1.1. Correspondingly, the *outer parallel set* at distance $t \geq 0$ is the set

$$\{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega) \leq t\} = \Omega + tB,$$

where $+$ denotes the *Minkowski sum*. In this paper we provide a lower bound for the perimeter of Ω_t in terms of the perimeter of Ω.

An important result in the theory of outer parallel sets is the so-called *Steiner formula*

$$|\Omega + tB| = \sum_{i=0}^{n} \binom{n}{i} t^i W_i(\Omega), \tag{1}$$

where coefficients W_i of the polynomial are the *quermassintegrals* of Ω, which are a special case of mixed volumes (see Section 1.1). The set of quermassintegrals contains several important geometric quantities: for instance we have that $W_0(\Omega) = |\Omega|$ and $nW_1(\Omega) = |\partial\Omega|$. There are analogous formulae to (1), called the *Steiner formulae* [14], that express the value of the i-th quermassintegral of $\Omega + tB$ in terms of $W_j(\Omega)$, for $i \leq j \leq n$. The Steiner formula appears not only in convex geometry, and important applications may be found in Federer's work on curvature measures in geometric measure theory (see [5]) and Weyl's tube formula in differential geometry (see [17]).

For inner parallel sets there is, in general, no counterpart to the Steiner formula. Matheron conjectured in [11] that the volume of a Minkowski difference is bounded from below by the *alternating Steiner polynomial.* If we restrict our attention to inner parallel sets he conjectured that

$$|\Omega \sim tB| \geq \sum_{i=0}^{n} \binom{n}{i} (-t)^i W_i(\Omega).$$

The precise conjecture was a more general statement where B is replaced by a general convex body and the quermassintegrals are replaced by mixed volumes. However, the conjecture was proved to be false by Hernández Cifre and Saorín in [7].

In addition to the lack of a Steiner-type formula, the Minkowski difference is far from being as well behaved as the Minkowski sum. In contrast to the Minkowski sum the difference is not a vectorial operation. Moreover, the regularity properties of $\Omega \sim tB$ may be very different from those of Ω. Both of these properties are demonstrated in Fig. 1. Nonetheless, the theory of inner parallel sets is rich and has several beautiful applications in both convex geometry and analysis (see for instance [2,4,10,12,13]).

In [8] the authors prove bounds for the quermassintegrals of inner parallel sets in a more general setting than that described above. Instead of considering the sets $\Omega \sim tB$, $t \geq 0$, they consider $\Omega \sim tE$ for some convex set E. The inequalities obtained in this paper are closely related to those in [8], and using similar techniques the results here could, at least in some sense, be generalized to the same setting. However, in such