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Local stability of the free additive convolution

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We prove that the system of subordination equations, defining the free additive convolution of two probability measures, is stable away from the edges of the support and blow-up singularities by showing that the recent smoothness condition of Kargin is always satisfied. As an application, we consider the local spectral statistics of the random matrix ensemble $A+UBU^*$, where U is a Haar distributed random unitary or orthogonal matrix, and A and B are deterministic matrices. In the bulk regime, we prove that the empirical spectral distribution of $A+UBU^*$ concentrates around the free additive convolution of the spectral distributions of A and B on scales down to $N^{-2/3}$.

1. Introduction

One of the basic concepts of free probability theory is the free additive convolution of two probability laws in a non-commutative probability space; it describes the law of the

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sum of two free random variables. In the case of a bounded self-adjoint random variable, its law can be identified with a probability measure of compact support on the real line. Hence the free additive convolution of two probability measures is a well-defined concept and it is characteristically different from the classical convolution.

In this paper, we prove a local stability result of the free additive convolution. A direct consequence is the continuity of the free additive convolution in a much stronger topology than established earlier by Bercovici and Voiculescu [10]. A second application of our stability result is to establish a local law on a very small scale for the eigenvalue density of a random matrix ensemble $A + UBU^*$ where U is a Haar distributed unitary or orthogonal matrix and A, B are deterministic N by N hermitian matrices.

The free additive convolution was originally introduced by Voiculescu [36] for the sum of free bounded noncommutative random variables in an algebraic setup (see Maassen [32] and by Bercovici and Voiculescu [10] for extensions to the unbounded case). The Stieltjes transform of the free additive convolution is related to the Cauchy–Stieltjes transforms of the original measures by an elegant analytic change of variables. This *subordination phenomenon* was first observed by Voiculescu [38] in a generic situation and extended to full generality by Biane [14]. In fact, the subordination equations, see (2.5)–(2.6) below, may directly be used to define the free additive convolution. This analytic definition was given independently by Belinschi and Bercovici [7] and by Chistyakov and Götze [18]; for further details we refer to, *e.g.,* [39,27,2].

Kargin [30] pointed out that the analytic approach to the subordination equations, in contrast to the algebraic one, allows one to effectively study how free additive convolution is affected by small perturbations; this is especially useful to treat various error terms in the random matrix problem [31]. The basic tool is a local stability analysis of the subordination equations. In [30], Kargin assumed a lower bound on the imaginary part of the subordination functions and a certain non-degeneracy condition on the Jacobian that holds for generic values of the spectral parameter. While these so-called *smoothness conditions* hold in many examples, a general characterization was lacking. Our first result, Theorem 2.5, shows that the smoothness conditions hold wherever the absolutely continuous part of the free convolution measure is finite and nonzero. In particular, local stability holds unconditionally (Corollary 2.6) and, following Kargin's argument [30], we immediately obtain the continuity of the free additive convolution in a stronger sense; see Theorem 2.7.

The random matrix application of this stability result, however, goes well beyond Kargin's analysis [31] since our proof is valid on a much smaller scale. To explain the new elements, we recall how free probability connects to random matrices.

The following fundamental observation was made by Voiculescu [37] (later extended by Dykema [20] and Speicher [35]): if $A = A^{(N)}$ and $B = B^{(N)}$ are two sequences of Hermitian matrices that are asymptotically free with eigenvalue distributions converging to probability measures μ_α and μ_β, then the eigenvalue density of $A+B$ is asymptotically given by the free additive convolution $\mu_\alpha \boxplus \mu_\beta$. One of the most natural ways to ensure