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Blowup for fractional NLS

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ABSTRACT

We consider fractional NLS with focusing power-type nonlinearity

$$i\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $1/2 < s < 1$ and $0 < \sigma < \infty$ for $s \geqslant N/2$ and $0 < \sigma \leqslant 2s/(N-2s)$ for $s < N/2$. We prove a general criterion for blowup of radial solutions in $\mathbb{R}^N$ with $N \geqslant 2$ for L^2-supercritical and L^2-critical powers $\sigma \geqslant 2s/N$. In addition, we study the case of fractional NLS posed on a bounded star-shaped domain $\Omega \subset \mathbb{R}^N$ in any dimension $N \geqslant 1$ and subject to exterior Dirichlet conditions. In this setting, we prove a general blowup result without imposing any symmetry assumption on $u(t,x)$.

For the blowup proof in $\mathbb{R}^N$, we derive a localized virial estimate for fractional NLS in $\mathbb{R}^N$, which uses Balakrishnan's formula for the fractional Laplacian $(-\Delta)^s$ from semigroup theory. In the setting of bounded domains, we use a Pohozaev-type estimate for the fractional Laplacian to prove blowup.

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1. Introduction and main results

In this paper, we derive general criteria for blowup of solutions $u = u(t, x)$ for fractional NLS with focusing power-type nonlinearity given by

$$i\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.1}$$

Here the integer $N \geqslant 1$ denotes the space dimension, $(-\Delta)^s$ stands for the fractional Laplacian with power $s \in (0, 1)$, defined by its symbol $|\xi|^{2s}$ in Fourier space, and $\sigma > 0$ is a given exponent. The evolution problem (1.1) can be seen as a canonical model for a nonlocal dispersive PDE with focusing nonlinearity that can exhibit solitary waves, turbulence phenomena, and blowup of solutions (i.e. singularity formation). We refer to [4,32,19,22,16,5,6,21] for a (non-exhaustive) list of studies of fractional NLS in mathematics, numerics, and physics.

Although problem (1.1) bears a strong resemblance to the well-studied classical NLS (corresponding to $s = 1$), a general existence theorem for blowup solutions of problem (1.1) has remained a challenging open problem so far. To the best of the authors' knowledge, the cases that have been successfully addressed by now are: i) fractional NLS with nonlocal Hartree-type nonlinearities and radial data [12,5], and ii) a perturbative construction of minimal mass blowup solutions for the so-called focusing half-wave equation in $N = 1$ dimension [22]. Despite these efforts, the existence of blowup solutions for the model case of fractional NLS with power-type nonlinearity has mainly remained elusive up to now, but it has been strongly supported by numerical evidence [21]. In the present paper, we derive general blowup results for (1.1) in both the L^2-supercritical and L^2-critical cases where $\sigma > 2s/N$ and $\sigma = 2s/N$, respectively. In what follows, we shall discuss blowup for the fractional NLS (1.1) posed on all of $\mathbb{R}^N$ as well as on bounded domains. We treat these two cases separately as follows.

1.1. Radial blowup in $\mathbb{R}^N$

We consider the initial-value problem

$$\begin{cases} i\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u, \\ u(0, x) = u_0(x) \in H^s(\mathbb{R}^N), \quad u : [0, T) \times \mathbb{R}^N \to \mathbb{C}. \end{cases} \tag{fNLS}$$

Recall that we assume that $s \in (0, 1)$, $\sigma > 0$, and $N \geqslant 1$ denotes the space dimension. In what follows, we shall assume that we are given a sufficiently regular solution $u(t)$. More precisely, that $u \in C([0, T); H^{2s}(\mathbb{R}^N))$ for reasons explained below. Let us mention that the local well-posedness theory for the range of $s \in (0, 1)$, $N \geqslant 1$, and exponents $\sigma > 0$ considered below is not completely settled yet; see, e.g., [30,16] for local well-posedness results for non-radial and radial data, respectively.