Journal of Functional Analysis 270 (2016) 4443–4465

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Γ-supercyclicity

S. Charpentier [a,*], R. Ernst [b], Q. Menet [c]

[a] *Institut de Mathématiques, UMR 7373, Aix-Marseille Université, 39 rue F. Joliot Curie, 13453 Marseille Cedex 13, France*

[b] *Laboratoire Paul Painlevé, UMR 8524 et École Centrale de Lille, Cité Scientifique, CS20048, 59651 Villeneuve d'Ascq cedex, France*

[c] *Laboratoire de Mathématiques de Lens, Université d'Artois, Rue Jean Souvraz S.P. 18, 62307 Lens, France*

A R T I C L E   I N F O

Article history:
Received 15 September 2015
Accepted 3 March 2016
Available online 22 March 2016
Communicated by Stefaan Vaes

MSC:
47A16

Keywords:
Hypercyclicity
Supercyclicity

A B S T R A C T

We characterize the subsets Γ of $\mathbb{C}$ for which the notion of Γ-supercyclicity coincides with the notion of hypercyclicity, where an operator T on a Banach space X is said to be Γ-supercyclic if there exists $x \in X$ such that $\overline{\mathrm{Orb}}(\Gamma x, T) = X$. In addition we characterize the sets $\Gamma \subset \mathbb{C}$ for which, for every operator T on X, T is hypercyclic if and only if there exists a vector $x \in X$ such that the set $\mathrm{Orb}(\Gamma x, T)$ is somewhere dense in X. This extends results by León–Müller and Bourdon–Feldman respectively. We are also interested in the description of those sets $\Gamma \subset \mathbb{C}$ for which Γ-supercyclicity is equivalent to supercyclicity.

* Corresponding author.
E-mail addresses: stephane.charpentier.1@univ-amu.fr (S. Charpentier), ernst.r@math.cnrs.fr (R. Ernst), quentin.menet@univ-artois.fr (Q. Menet).

http://dx.doi.org/10.1016/j.jfa.2016.03.005

1. Introduction and statements of the main results

1.1. Introduction

Let X be a complex Banach space and let $L(X)$ denote the space of bounded linear operators on X. For T in $L(X)$, x in X, and Γ a non-empty subset of the complex plane $\mathbb{C}$, we denote $\mathrm{Orb}(\Gamma x, T) = \{\gamma T^n x : \gamma \in \Gamma,\ n \geq 0\}$. We say that x is Γ-*supercyclic* for T if $\mathrm{Orb}(\Gamma x, T)$ is dense in X and T will be said to be Γ-supercyclic if it admits a Γ-supercyclic vector. In particular, if $\Gamma = \mathbb{C}$, x Γ-supercyclic for T reads x supercyclic for T and if Γ reduces to a single nonzero point, x Γ-supercyclic for T reads x hypercyclic for T. The notion of hypercyclicity was already studied by Birkhoff in the twenties but it really began to attract much attention in the late seventies. The terminology follows that of supercyclicity, introduced by Hilden and Wallen [19] in the early seventies, and the former notion of cyclicity. While the latter is directly connected with the well-known Invariant Subspace Problem, hypercyclicity is connected with the Invariant *Subset* Problem. We refer to the books [4,17] for a deep introduction to linear dynamics.

One of the first important results was Kitai Criterion [20], refined by Bès [7] in the following form and known as the Hypercyclicity Criterion.

Theorem *(Hypercyclicity criterion). Let $T \in L(X)$. We assume that there exist two dense subsets $X_0, Y_0 \subset X$, an increasing sequence $(n_k)_k \subset \mathbb{N}$, and maps $S_{n_k} : Y_0 \to X$ such that for any $x \in X_0$ and $y \in Y_0$ the following holds:*

(1) $T^{n_k} x \to 0$ as $k \to \infty$;
(2) $S_{n_k} y \to 0$ as $k \to \infty$;
(3) $T^{n_k} S_{n_k} y \to y$ as $k \to \infty$.

Then T is hypercyclic.

We mention that there also exists a so-called Supercyclicity Criterion due to Salas [28], which is easily seen to be non-necessary for supercyclicity. The Hypercyclicity Criterion gives an effective way of proving that an operator is hypercyclic and covers a wide range of *concrete* hypercyclic operators, allowing to directly recover historical examples of hypercyclic operators exhibited by Birkhoff [9], MacLane [23] or Rolewicz [27]. A long-standing major open question was to know whether the Hypercyclicity Criterion is necessary for an operator to be hypercyclic. Bès and Peris [8] observed that satisfying the Hypercyclicity Criterion is in fact equivalent to being hereditarily hypercyclic or weakly mixing and, in 2008, De La Rosa and Read built a Banach space and a non-weakly mixing hypercyclic operator acting on this space, giving a negative answer to the above mentioned question. A bit later, Bayart and Matheron [3] provided such an example on many classical Banach spaces, including the separable Hilbert space. We also