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# A criterion for integrability of matrix coefficients with respect to a symmetric space ☆

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ABSTRACT

Let $G$ be a reductive group and $\theta$ an involution on $G$, both defined over a $p$-adic field. We provide a criterion for $G^\theta$-integrability of matrix coefficients of representations of $G$ in terms of their exponents along $\theta$-stable parabolic subgroups. The group case reduces to Casselman's square-integrability criterion. As a consequence we assert that certain families of symmetric spaces are strongly tempered in the sense of Sakellaridis and Venkatesh. For some other families our result implies that matrix coefficients of all irreducible, discrete series representations are $G^\theta$-integrable.



**Contents**



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## 1. Introduction

Let $F$ be a $p$-adic field. Let $G$ be the group of $F$-points of a reductive $F$-group, $\theta$ an involution on $G$ and $H = G^\theta$ the subgroup of $\theta$-fixed points. In this work we provide a criterion for $H$-integrability of matrix coefficients of admissible representations of $G$ in terms of their exponents along $\theta$-stable parabolic subgroups of $G$. In the group case ($G = H \times H$, $\theta(x, y) = (y, x)$) our result reduces to Casselman's square-integrability criterion [6, Theorem 4.4.6].

For a smooth representation $\pi$ of $G$, let $\mathrm{Hom}_H(\pi, \mathbb{C})$ be the space of $H$-invariant linear forms on $\pi$. As is apparent, for example, from the general treatment of [2], this space plays an essential role in the harmonic analysis of the space $G/H$. See also [5] for the study of $H$-invariant linear forms on induced representations in the context of a $p$-adic symmetric space and [15] in the more general setting of a spherical variety.

Furthermore, the understanding of $H$-invariant linear forms in the local setting has applications to the study of period integrals of automorphic forms. A conjecture of Ichino–Ikeda [9] treats a different setting in which the pair $(G, H)$ is of the Gross–Prasad type. It claims, roughly speaking, that under appropriate assumptions, the Hermitian form on an irreducible, tempered, automorphic representation of $G$ associated to the absolute value squared of the $H$-period integral factorizes as a product of local $H$-integrals of the associated matrix coefficients. The conjectural framework of [15] suggests a generalization of this phenomenon, which will include the symmetric case. (For an explicit factorization of a somewhat different nature see e.g. [10,7].)

Integrability of matrix coefficients provides an explicit construction of the local components of period integrals of automorphic forms. Factorizable period integrals, in turn, are intimately related with special values of $L$-functions and with Langlands functoriality conjectures.

The above global conjectures suggest to study the following purely local questions. Let $A_G$ be the maximal split torus in the center of $G$ and $A_G^+$ the connected component of its intersection with $H$. Let $\pi$ be a smooth representation of $G$ and $\tilde{v}$ a smooth linear form in its contragredient $\tilde{\pi}$.

- Is the linear form

$$\ell_{\tilde{v},H}(v) := \int_{H/A_G^+} \tilde{v}(\pi(h)v)\, dh$$

  well defined on $\pi$ by an absolutely convergent integral? (In this case $\ell_{\tilde{v},H} \in \mathrm{Hom}_H(\pi, \mathbb{C})$.)
- Is it non-zero?

The answer we provide for the first question is a relative analogue of Casselman's criterion. We recall that, essentially, that criterion says that an admissible representation