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Φ-moment inequalities for independent and freely independent random variables

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A B S T R A C T

This paper is devoted to the study of Φ-moments of sums of independent/freely independent random variables. More precisely, let $(f_k)_{k=1}^n$ be a sequence of positive (symmetrically distributed) independent random variables and let Φ be an Orlicz function with Δ_2-condition. We provide an equivalent expression for the quantity $\mathbb{E}(\Phi(\sum_{k=1}^n f_k))$ in term of the sum of disjoint copies of the sequence $(f_k)_{k=1}^n$. We also prove an analogous result in the setting of free probability. Furthermore, we provide an equivalent characterization of $\tau(\Phi(\sup^+_{1\le k\le n} x_k))$ for positive freely independent random variables and also present some new results on free Johnson–Schechtman inequalities in the quasi-Banach symmetric operator space.

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1. Introduction

The main theme of this article is twofold: it concerns Φ-moment estimates of independent random variables and of freely independent self-adjoint operators affiliated with a finite von Neumann algebra.

In order to explain the classical probability theory roots of our study, we need to recall two outstanding results published simultaneously in 1970, due to Kruglov [32] and Rosenthal [41], respectively. The results in [32] were concerned with infinitely divisible distributions occurring in the analysis of the classical Levy–Khintchine formula. Let f be a random variable on $(0,1)$, and let $\pi(f)$ denote the random variable $\sum_{k=1}^{N} f_k$, where f_k, $k \geq 1$, are independent copies of f and N is a Poisson random variable independent from the sequence (f_k). In [32], the following Φ-moment theorem was proved.

Theorem 1.1 *(Kruglov Theorem). Suppose that Φ is a positive continuous function on $\mathbb{R}$ with $\Phi(0) = 0$ and suppose that it satisfies one of the following conditions.*

(i) $\Phi(t+s) \leq B\Phi(t)\Phi(s)$ *for every* $s, t \in \mathbb{R}$ *and some constant* $B > 0$.
(ii) $\Phi(t+s) \leq B(\Phi(t) + \Phi(s))$ *for every* $s, t \in \mathbb{R}$ *and some constant* $B > 0$.

For an arbitrary random variable f the condition $\mathbb{E}(\Phi(\pi(f))) < \infty$ is equivalent to the condition $\mathbb{E}(\Phi(f)) < \infty$.

The Rosenthal theorem [41] concerns only the special case $\Phi(t) = |t|^p$, $p \geq 1$ and was established while studying the L_p-norm of a sum of independent functions.

Theorem 1.2 *(Rosenthal Theorem). If $p > 2$, then there exists a constant $K_p > 0$ such that for an arbitrary sequence $\{f_k\}_{k=1}^{\infty} \subset L_p$ of independent functions satisfying $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ and for every $n \geq 1$ the following estimates hold:*

$$\frac{1}{2}\max\left\{\left(\sum_{k=1}^{n}\|f_k\|_p^p\right)^{1/p}, \left(\sum_{k=1}^{n}\|f_k\|_2^2\right)^{1/2}\right\} \leq \left\|\sum_{k=1}^{n} f_k\right\|_p \leq$$

$$\leq K_p \max\left\{\left(\sum_{k=1}^{n}\|f_k\|_p^p\right)^{1/p}, \left(\sum_{k=1}^{n}\|f_k\|_2^2\right)^{1/2}\right\}. \tag{1.1}$$

These two seemingly disconnected results, are in fact deeply connected. To explain better this connection, we need to refer to works of several other mathematicians. For convenience, let us introduce the notation

$$\bigoplus_{k=1}^{n} f_k := \sum_{k=1}^{n} f_k(\cdot - k + 1)\chi_{[k-1,k)}$$

for disjoint sum of random variables (f_k) on $(0,1)$, which is a Lebesgue measurable function on $(0,\infty)$. Then Theorem 1.2 can be restated as