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Idempotents with small norms

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A R T I C L E   I N F O

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A B S T R A C T

Let Γ be a locally compact group. We answer two questions left open in [8] and [10]:

(i) For abelian Γ, we prove that if $\chi_S \in \mathrm{B}(\Gamma)$ is an idempotent with norm $\|\chi_S\| < \frac{4}{3}$, then S is the union of two cosets of an open subgroup of Γ.
(ii) For general Γ, we prove that if $\chi_S \in M_{cb}\mathrm{A}(\Gamma)$ is an idempotent with norm $\|\chi_S\|_{cb} < \frac{1+\sqrt{2}}{2}$, then S is an open coset in Γ.

1. Introduction

In his 1968 papers, Saeki determined idempotent measures on a locally compact abelian group G with small norms. This is equivalent to determining idempotent functions in the Fourier–Stieltjes algebras $\mathrm{B}(\Gamma)$ on a locally compact abelian group Γ with

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small norms (where Γ and G could be taken as Pontryagin duals of each other). The statements of Saeki's results in the Fourier–Stieltjes setting are:

Theorem 1.1 *(Saeki). Let Γ be a locally compact abelian group, and let φ be an idempotent function in $\mathrm{B}(\Gamma)$ so that $\varphi = \chi_S$ for some nonempty $S \subseteq \Gamma$. Then:*

(i) *(See [7].) If $\|\varphi\| < \frac{1+\sqrt{2}}{2}$, then S is an open coset in Γ.*
(ii) *(See [8].) If $\|\varphi\| \in (1, \frac{\sqrt{17}+1}{4})$, then S is the union of two cosets of an open subgroup of Γ but is not a coset itself.*

For abelian Γ, it is well-known (see [6, p. 73]) that if S is an open coset in Γ, then $\|\chi_S\| = 1$, and whereas if S is the union of two cosets of an open subgroup of Γ but is not a coset itself, then

$$\|\chi_S\| = \begin{cases} \frac{2}{q\sin(\pi/2q)} & \text{if } q \text{ is odd} \\ \frac{2}{q\tan(\pi/2q)} & \text{if } q \text{ is even} \\ \frac{4}{\pi} & \text{if } q = \infty \end{cases} \tag{1.1}$$

where $q \geq 3$ is the "relative order" of the two cosets forming S. The largest value in (1.1) is $\frac{4}{3}$ when $q = 3$ and the smallest one is $\frac{1+\sqrt{2}}{2}$ when $q = 4$. In particular, the number $\frac{1+\sqrt{2}}{2}$ in Theorem 1.1(i) is sharp.

The paper [8] asked whether or not the interval $(1, \frac{\sqrt{17}+1}{4})$ in Theorem 1.1(ii) could be increased to $(1, \frac{4}{3})$, and we answer this in affirmative in Theorem 3.4. Note that the interval $(1, \frac{4}{3})$ is sharp because of the discussion in the previous paragraph and also since there are idempotents χ_S of $\mathrm{B}(\Gamma)$ with $\|\chi_S\| = \frac{4}{3}$ but S is not any union of two cosets of open subgroup of Γ (see the last paragraph of [8]).

Lesser is known about idempotents in $\mathrm{B}(\Gamma)$ with small norms for general locally compact group Γ. Ilie and Spronk [4] proved that χ_S is an idempotent in $\mathrm{B}(\Gamma)$ with $\|\chi_S\| = 1$ if and only if S is an open coset in Γ; the Fourier–Stieltjes norm is weakened to the norm $\|\cdot\|_{cb}$ of $M_{cb}\mathrm{A}(\Gamma)$ by Forrest and Runde in [3]. More generally, Stan proved the following.

Theorem 1.2. *(See Stan [10].) Let Γ be a locally compact group, and let φ be an idempotent function in $M_{cb}\mathrm{A}(\Gamma)$ so that $\varphi = \chi_S$ for some nonempty $S \subseteq \Gamma$. If $\|\varphi\|_{cb} < \frac{2}{\sqrt{3}}$, then S is an open coset in Γ, and in which case $\|\varphi\|_{cb} = 1$.*

Here $M_{cb}\mathrm{A}(\Gamma)$ is the completely bounded multiplier algebra of the Fourier algebra $\mathrm{A}(\Gamma)$ and is defined as follows. Since the Fourier algebra $\mathrm{A}(\Gamma)$ is the predual of the group von Neumann algebra $\mathrm{VN}(\Gamma)$, it has a canonical operator space structure, which makes it a completely contractive operator algebra (see the monograph [2] for more details). The completely bounded multiplier algebra $M_{cb}\mathrm{A}(\Gamma)$ of $\mathrm{A}(\Gamma)$ consists of those continuous functions $\varphi : \Gamma \to \mathbb{C}$ such that the mapping