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On variational characterization of four-end solutions of the Allen–Cahn equation in the plane

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A B S T R A C T

In this paper a novel variational method is developed to construct four-end solutions in $\mathbb{R}^2$ for the Allen–Cahn equation. Four-end solutions have been constructed by Del Pino, Kowalczyk, Pacard and Wei when the angle θ of the ends is close to $\pi/2$ or 0 using the Lyapunov–Schmid reduction method, and later by Kowalczyk, Liu and Pacard using a continuation method for general $\theta \in (0, \pi/2)$. By a special mountain pass argument in a restricted space, namely a set of monotone paths of monotone functions, a family of solutions in bounded domain is constructed with very good control of their nodal sets, which then leads to a four-end solution after sending the domains to $\mathbb{R}^2$. In this way, not only a family of four end solutions is constructed for any angle $\theta \in (0, \pi/2)$. The Morse index of such a four-end solution is also shown to be one.

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1. Introduction

In this paper, we will study the four-end solutions to the following Allen–Cahn equation in the plane:

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^2, \ |u| < 1. \tag{1}$$

By definition, a solution to (1) is called a $2k$-end solution, if its nodal lines are asymptotic to $2k$ half straight lines at infinity ([7]). Recent progresses indicate that the class of multiple-end solutions are abundant. It is also expected that multiple-end solutions are equivalent to finite Morse index solutions, although up to now no rigorous proof has been given. The one dimensional heteroclinic solutions $\tanh\left(\frac{a_1 x + a_2 y + b}{\sqrt{2}}\right)$, where $a_1^2 + a_2^2 = 1$, are the simplest examples of two-end solutions. Minimizing arguments (Alessio–Calamai–Piero Montecchiari [2], Dang–Fife–Peletier [5]) show that for each number $k \in \mathbb{N}$, there is a $2k$-end solution with dihedral symmetry whose nodal lines are exactly k straight lines intersecting at the origin with consecutive angle $\frac{\pi}{k}$. When $k = 2$, it is the well known saddle solution. There are also multiple-end solutions without symmetry. Indeed, using infinite dimensional Lyapunov–Schmidt reduction method, a family of $2k$-end solutions has been constructed by del Pino–Kowalczyk–Pacard–Wei ([7]), the nodal lines of these solutions are close to the solutions of finite non-periodic Toda system and consist of k almost parallel curved lines. Concerning the structure of the set of solutions, one knows from a result by del Pino–Kowalczyk–Pacard ([6]) that the moduli space of $2k$-end solutions has the structure of a real analytic variety of formal dimension $2k$. The above mentioned solutions arising from Toda system could be regarded as located near the "boundary" of this moduli space. The end-to-end construction performed in [17] then yields another family of multiple-end solutions near the boundary of the moduli space. The basic blocks for the end-to-end construction are four-end solutions which will be described in the next paragraph. Indeed, one expects that these four-end solutions should also be able to be used to construct other interesting solutions, not only in the plane, but also in higher dimensional Euclidean spaces.

Next let us recall some known results concerning four-end solutions. By a result of Gui ([10]), after a possible rigid motion of the coordinate system, any four-end solution is even with respect to the x and y axis. Moreover, multiplying the solution u by -1 if necessary, it has the following monotonicity property:

$$\partial_x u < 0 \text{ for } x > 0; \ \partial_y u > 0 \text{ for } y > 0.$$

Then in the first quadrant, the nodal set of u is asymptotic to certain straight line $y = x \tan\theta + b$. We shall call θ the angle of u and denote it by $\Theta(u)$. It has been proven by Kowalczyk–Liu–Pacard ([16]) using continuation method that the set of four-end solutions, modulo rigid motions of the plane, is diffeomorphic to the open interval $(0, 1)$ in a natural introduced topology on the moduli spaces. Now let us point out that the