Journal of Functional Analysis 271 (2016) 2701–2751

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

# On the definition and the properties of the principal eigenvalue of some nonlocal operators

Henri Berestycki [a], Jérôme Coville [b,*], Hoang-Hung Vo [c]

[a] CAMS, École des Hautes Études en Sciences Sociales, 190-198 avenue de France, 75013, Paris, France
[b] UR 546 Biostatistique et Processus Spatiaux, INRA, Domaine St Paul Site Agroparc, F-84000 Avignon, France
[c] Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City National University, No. 227 Nguyen Van Cu Street, Ward 4, District 5, Ho Chi Minh City, Vietnam

A R T I C L E I N F O

*Article history:*
Received 25 February 2016
Accepted 25 May 2016
Available online 2 June 2016
Communicated by B. Schlein

*Keywords:*
Principal eigenvalue
Non-local operator
Definition and variational characterisation
Singular limits

A B S T R A C T

In this article we study some spectral properties of the linear operator $\mathcal{L}_\Omega + a$ defined on the space $C(\bar\Omega)$ by:

$$\mathcal{L}_\Omega[\varphi] + a\varphi := \int_\Omega K(x,y)\varphi(y)\,dy + a(x)\varphi(x)$$

where $\Omega \subset \mathbb{R}^N$ is a domain, possibly unbounded, $a$ is a continuous bounded function and $K$ is a continuous, non-negative kernel satisfying an integrability condition.
We focus our analysis on the properties of the generalised principal eigenvalue $\lambda_p(\mathcal{L}_\Omega + a)$ defined by

$$\lambda_p(\mathcal{L}_\Omega + a) := \sup\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\bar\Omega), \varphi > 0, \quad \textit{such that} \;\; \mathcal{L}_\Omega[\varphi] + a\varphi + \lambda\varphi \le 0 \;\text{ in }\; \Omega\}.$$

We establish some new properties of this generalised principal eigenvalue $\lambda_p$. Namely, we prove the equivalence of different definitions of the principal eigenvalue. We also study the behaviour of $\lambda_p(\mathcal{L}_\Omega + a)$ with respect to some scaling of $K$.

* Corresponding author.
*E-mail addresses:* hb@ehess.fr (H. Berestycki), jerome.coville@avignon.inra.fr (J. Coville), vhhungkhtn@gmail.com (H.-H. Vo).

http://dx.doi.org/10.1016/j.jfa.2016.05.017

For kernels $K$ of the type, $K(x,y) = J(x-y)$ with $J$ a compactly supported probability density, we also establish some asymptotic properties of $\lambda_p\left(\mathcal{L}_{\sigma,m,\Omega} - \frac{1}{\sigma^m} + a\right)$ where $\mathcal{L}_{\sigma,m,\Omega}$ is defined by $\mathcal{L}_{\sigma,2,\Omega}[\varphi] := \frac{1}{\sigma^{2+N}} \int_\Omega J\left(\frac{x-y}{\sigma}\right)\varphi(y)\,dy$. In particular, we prove that

$$\lim_{\sigma\to 0} \lambda_p\left(\mathcal{L}_{\sigma,2,\Omega} - \frac{1}{\sigma^2} + a\right) = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right),$$

where $D_2(J) := \int_{\mathbb{R}^N} J(z)|z|^2\,dz$ and $\lambda_1$ denotes the Dirichlet principal eigenvalue of the elliptic operator. In addition, we obtain some convergence results for the corresponding eigenfunction $\varphi_{p,\sigma}$.



**Contents**



## 1. Introduction

The principal eigenvalue of an operator is a fundamental notion in modern analysis. In particular, this notion is widely used in PDE's literature and is at the source of many profound results especially in the study of elliptic semi-linear problems. For example, the principal eigenvalue is used to characterise the stability of equilibrium of a reaction–diffusion equation enabling the definition of persistence criteria [18–20,5,33,44,53]. It is also an important tool in the characterisation of maximum principle properties satisfied by elliptic operators [12,8] and to describe continuous semi-groups that preserve an or-