Journal of Functional Analysis 271 (2016) 2752–2775

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Bilinear Kakeya–Nikodym averages of eigenfunctions on compact Riemannian surfaces [☆]

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A R T I C L E I N F O

Article history:
Received 26 February 2016
Accepted 12 April 2016
Available online 22 April 2016
Communicated by L. Saloff-Coste

Keywords:
Eigenfunctions
Bilinear estimates
Kakeya–Nikodym averages

A B S T R A C T

We obtain an improvement of the bilinear estimates of Burq, Gérard and Tzvetkov [6] in the spirit of the refined Kakeya–Nikodym estimates [2] of Blair and the second author. We do this by using microlocal techniques and a bilinear version of Hörmander's oscillatory integral theorem in [7].

☆ The first author was supported in part by the NSF of China (No. 11231006, No. 11371059). The second author was supported in part by the NSF grant DMS-1361476. The fourth author was supported by ERC Advanced Grant No. 291214 BLOWDISOL.

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http://dx.doi.org/10.1016/j.jfa.2016.04.015

1. Introduction

Let (M, g) be a two-dimensional compact boundaryless Riemannian manifold with Laplacian Δ_g. If e_λ are the associated eigenfunctions of $\sqrt{-\Delta_g}$ such that $-\Delta_g e_\lambda = \lambda^2 e_\lambda$, then it is well known that

$$\|e_\lambda\|_{L^4(M)} \le C\,\lambda^{\frac{1}{8}}\,\|e_\lambda\|_{L^2(M)}, \tag{1.1}$$

which was proved in [9] using approximate spectral projectors $\chi_\lambda = \chi(\lambda - \sqrt{-\Delta_g})$ and showing

$$\|\chi_\lambda f\|_{L^4(M)} \le C\,\lambda^{\frac{1}{8}}\,\|f\|_{L^2(M)}. \tag{1.2}$$

If $0 < \lambda \le \mu$ and e_λ, e_μ are two associated eigenfunctions of $\sqrt{-\Delta_g}$ as above, Burq et al. [6] proved the following bilinear L^2-refinement of (1.1)

$$\|e_\lambda e_\mu\|_{L^2(M)} \le C\,\lambda^{\frac{1}{4}}\,\|e_\lambda\|_{L^2(M)}\|e_\mu\|_{L^2(M)}, \tag{1.3}$$

as a consequence of a more general bilinear estimate on the reproducing operators

$$\big\|\chi_\lambda f \chi_\mu g\big\|_{L^2(M)} \le C\,\lambda^{\frac{1}{4}}\,\|f\|_{L^2(M)}\|g\|_{L^2(M)}. \tag{1.4}$$

The bilinear estimate (1.3) plays an important role in the theory of nonlinear Schrödinger equations on compact Riemannian surfaces and it is sharp in the case when $M = \mathbb{S}^2$ endowed with the canonical metric and $e_\lambda(x) = h_p(x)$, $e_\mu(x) = h_q(x)$ are highest weight spherical harmonic functions of degree p and q, concentrating along the equator

$$\big\{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, x_3 = 0\big\}$$

with $\lambda^2 = p(p+1)$, $\mu^2 = q(q+1)$. Indeed, one may take $h_k(x) = (x_1 + ix_2)^k$ to see $\|h_k\|_2 \approx k^{-1/4}$ by direct computation.

In Section 2, we will construct a generic example to show the optimality of (1.4) and exhibit that the mechanism responsible for the optimality seems to be the existence of eigenfunctions concentrating along a tubular neighborhood of a segment of a geodesic. As observed in [10], (1.2) is saturated by constructing an oscillatory integral which highly concentrates along a geodesic. The dynamical behavior of geodesic flows on M accounts for the analytical properties of eigenfunctions exhibits the transference of mathematical theory from classical mechanics to quantum mechanics (see [12]).

That the eigenfunctions concentrating along geodesics yield sharp spectral projector inequalities leads naturally to the refinement of (1.1) in [5] and [11], where it is proved for an L^2 normalized eigenfunction e_λ, its L^4-norm is essentially bounded by a power of

$$\sup_{\gamma\in\Pi} \frac{1}{|\,T_{\lambda^{-1/2}}(\gamma)\,|} \int\limits_{T_{\lambda^{-1/2}}(\gamma)} |\,e_\lambda(x)\,|^2\,dx, \tag{1.5}$$