Journal of Functional Analysis 271 (2016) 2808–2821

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Spectrality and tiling by cylindric domains ☆

Rachel Greenfeld, Nir Lev *

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

ARTICLE INFO

Article history:
Received 3 March 2016
Accepted 18 April 2016
Available online 2 May 2016
Communicated by Stefaan Vaes

Keywords:
Fuglede's conjecture
Spectral set
Tiling

ABSTRACT

A bounded set $\Omega \subset \mathbb{R}^d$ is called a spectral set if the space $L^2(\Omega)$ admits a complete orthogonal system of exponential functions. We prove that a cylindric set Ω is spectral if and only if its base is a spectral set. A similar characterization is obtained of the cylindric sets which can tile the space by translations.

1. Introduction

1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive Lebesgue measure. A discrete set $\Lambda \subset \mathbb{R}^d$ is called a *spectrum* for Ω if the system of exponential functions

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}, \quad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle}, \tag{1.1}$$

constitutes an orthogonal basis in $L^2(\Omega)$, that is, the system is orthogonal and complete in the space. A set Ω which admits a spectrum Λ is called a *spectral set*. For example, if Ω is the unit cube in $\mathbb{R}^d$, then it is a spectral set, and $\Lambda = \mathbb{Z}^d$ is a spectrum for Ω.

The study of spectral sets was initiated in the paper [1] due to Fuglede (1974), who conjectured that these sets could be characterized geometrically in the following way:

☆ Research partially supported by the Israel Science Foundation grant No. 225/13.

* Corresponding author.
E-mail addresses: rachelgrinf@gmail.com (R. Greenfeld), levnir@math.biu.ac.il (N. Lev).

http://dx.doi.org/10.1016/j.jfa.2016.04.021
0022-1236/

the set Ω *is spectral if and only if it can tile the space by translations.* We say that Ω *tiles* the space by translations along a discrete set $\Lambda \subset \mathbb{R}^d$ if the family of sets $\Omega + \lambda$ ($\lambda \in \Lambda$) constitutes a partition of $\mathbb{R}^d$ up to measure zero. Fuglede's conjecture inspired extensive research over the years, and a number of interesting results supporting the conjecture had been obtained.

For example, it was proved in [1] that if Ω tiles the space by translations along a *lattice*, then it is a spectral set. To the contrary, a triangle in the plane [1], or more generally, any convex non-symmetric domain in $\mathbb{R}^d$ [9], is not spectral. It was also proved that the ball in $\mathbb{R}^d$ ($d \geqslant 2$) is not a spectral set [1,4,2], as well as any convex domain with a smooth boundary [5]. In [6] it was proved that a convex domain $\Omega \subset \mathbb{R}^2$ is spectral if and only if it is either a parallelogram or a centrally symmetric hexagon, which confirmed that Fuglede's conjecture is true for convex domains in dimension $d = 2$. See also the survey in [11, Section 3].

On the other hand, in 2004 a counter-example to the "spectral implies tiling" part of the conjecture in dimensions $d \geqslant 5$ was found by Tao [15]. Subsequently, the "tiling implies spectral" part was also disproved, and the dimension in these counter-examples (all of which are finite unions of unit cubes) was reduced up to $d \geqslant 3$, see [12, Section 4] and the references given there. The conjecture is still open, though, in dimensions $d = 1, 2$ in both directions.

1.2. A bounded, measurable set $\Omega \subset \mathbb{R}^d$ ($d \geqslant 2$) will be called a *cylindric set* if it has the form

$$\Omega = I \times \Sigma, \tag{1.2}$$

where I is an interval in $\mathbb{R}$, and Σ is a measurable set in $\mathbb{R}^{d-1}$. In this case, the set Σ will be called the *base* of the cylindric set Ω.

In this paper we are interested in the spectrality problem for cylindric sets. For example, as far as we know, the following question has remained open: *Let* Ω *be a cylindric set in* $\mathbb{R}^d$ ($d \geqslant 3$)*, whose base* Σ *is the unit ball in* $\mathbb{R}^{d-1}$*. Is it a spectral set?*

As the boundary of this set Ω is not piecewise flat (and, in particular, Ω cannot tile), one would expect that the answer to this question should be negative. However the approach in [5] does not apply in this situation, as it is based on the existence of a point on the boundary of Ω where the Gaussian curvature is non-zero, while for a cylindric set this curvature vanishes at every point where the boundary is smooth.

The main result in this paper is the following:

Theorem 1.1. *A cylindric set* $\Omega = I \times \Sigma$ *is spectral (as a set in* $\mathbb{R}^d$, $d \geqslant 2$*) if and only if its base* Σ *is a spectral set (as a set in* $\mathbb{R}^{d-1}$*).*

Thus we obtain a characterization of the cylindric spectral sets Ω in terms of the spectrality of their base Σ.