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Variable weak Hardy spaces and their applications ☆

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Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition. In this article, the authors first introduce the variable weak Hardy space on $\mathbb{R}^n$, $WH^{p(\cdot)}(\mathbb{R}^n)$, via the radial grand maximal function, and then establish its radial or non-tangential maximal function characterizations. Moreover, the authors also obtain various equivalent characterizations of $WH^{p(\cdot)}(\mathbb{R}^n)$, respectively, by means of atoms, molecules, the Lusin area function, the Littlewood–Paley g-function or g_λ^*-function. As an application, the authors establish the boundedness of convolutional δ-type and non-convolutional γ-order Calderón–Zygmund operators from $H^{p(\cdot)}(\mathbb{R}^n)$ to $WH^{p(\cdot)}(\mathbb{R}^n)$ including the critical case when $p_- = n/(n+\delta)$ or when $p_- = n/(n+\gamma)$, where $p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$.

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1. Introduction

The main purpose of this article is to introduce and to investigate the variable weak Hardy spaces on $\mathbb{R}^n$. It is well known that the classical weak Hardy spaces appear naturally in critical cases of the study on the boundedness of operators. Indeed, the classical weak Hardy space $WH^1(\mathbb{R}^n)$ was originally introduced by Fefferman and Soria [18] when they tried to find out the biggest space from which the Hilbert transform is bounded to the weak Lebesgue space $WL^1(\mathbb{R}^n)$. Via establishing the ∞-atomic characterization of $WH^1(\mathbb{R}^n)$, they obtained the boundedness of some Calderón–Zygmund operators from $WH^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. Moreover, it is also well known that, when studying the boundedness of some singular integral operators, $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0,1]$; while when studying the boundedness of operators in the critical case, the Hardy spaces $H^p(\mathbb{R}^n)$ are usually further replaced by the weak Hardy space $WH^p(\mathbb{R}^n)$. For example, if $\delta \in (0,1]$ and T is a convolutional δ-type Calderón–Zygmund operator with $T^*(1) = 0$, where T^* denotes the *adjoint operator* of T, then T is bounded on $H^p(\mathbb{R}^n)$ for all $p \in (n/(n+\delta), 1]$ (see [5]), but may not be bounded on $H^{n/(n+\delta)}(\mathbb{R}^n)$. For such an endpoint case, Liu [30] proved that T is bounded from $H^{n/(n+\delta)}(\mathbb{R}^n)$ to $WH^{n/(n+\delta)}(\mathbb{R}^n)$ via establishing the ∞-atomic characterization of the weak Hardy space $WH^p(\mathbb{R}^n)$.

Furthermore, when studying the real interpolation between the Hardy space $H^p(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$, Fefferman et al. [17] proved that the weak Hardy spaces $WH^p(\mathbb{R}^n)$ also naturally appear as the intermediate spaces, which is another main motivation to develop a real-variable theory of $WH^p(\mathbb{R}^n)$. Recently, He [23] and Grafakos and He [22] further investigated vector-valued weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n, \ell^2)$ with $p \in (0,\infty)$. Very recently, Liang et al. [29] introduced a kind of generalized weak Hardy spaces of Musielak–Orlicz type $WH^\varphi(\mathbb{R}^n)$, which covers both weak Hardy spaces $WH^p(\mathbb{R}^n)$ and weighted weak Hardy spaces $WH^p_w(\mathbb{R}^n)$ from [34]. Various equivalent characterizations of $WH^\varphi(\mathbb{R}^n)$ by means of maximal functions, atoms, molecules and Littlewood–Paley functions, and the boundedness of Calderón–Zygmund operators in the critical case were obtained in [29]. For more related history and properties about $WH^p(\mathbb{R}^n)$, we refer the reader to [1,17,18,22,23,30,31,34] and their references.

On the other hand, based on the variable Lebesgue space, theories of several variable function spaces have rapidly been developed in recent years (see, for example, [3,4,12,14,32,41,44,45]). Recall that the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, with a variable exponent function $p(\cdot): \mathbb{R}^n \to (0,\infty)$, is a generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$, via replacing the constant exponent p by the exponent function $p(\cdot)$, which consists of all functions f such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx < \infty$. The study of variable Lebesgue spaces can be traced back to Orlicz [33], however, they have been the subject of more intensive study since the early 1990s because of their intrinsic interest for applications into harmonic analysis, partial differential equations and variational integrals with nonstandard growth conditions (see [10,13,26] and their references).