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Semitrivial vs. fully nontrivial ground states in cooperative cubic Schrödinger systems with $d \geq 3$ equations

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A B S T R A C T

In this work we consider the weakly coupled Schrödinger cubic system

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \mu_i u_i^3 + u_i \sum_{j \neq i} b_{ij} u_j^2 \\ u_i \in H^1(\mathbb{R}^N; \mathbb{R}), \quad i = 1, \ldots, d, \end{cases}$$

where $1 \leq N \leq 3$, $\lambda_i, \mu_i > 0$ and $b_{ij} = b_{ji} > 0$ for $i \neq j$. This system admits semitrivial solutions, that is solutions $\mathbf{u} = (u_1, \ldots, u_d)$ with null components. We provide optimal qualitative conditions on the parameters λ_i, μ_i and b_{ij} under which the *ground state solutions* have all components nontrivial, or, conversely, are semitrivial.
This question had been clarified only in the $d = 2$ equations case. For $d \geq 3$ equations, prior to the present paper, only very restrictive results were known, namely when the above system was a small perturbation of the super-symmetrical case $\lambda_i \equiv \lambda$ and $b_{ij} \equiv b$. We treat the general case, uncovering

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in particular a much more complex and richer structure with respect to the $d = 2$ case.

1. Introduction

We are interested in the elliptic system of d equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \mu_i u_i^3 + u_i \sum_{j\neq i} b_{ij} u_j^2 \\ u_i \in H^1(\mathbb{R}^N;\mathbb{R}), \quad i = 1,\dots,d, \end{cases} \tag{1.1}$$

in $\mathbb{R}^N$, $1 \leq N \leq 3$, with $\lambda_i, \mu_i > 0$ for every $i = 1,\dots,d$ and $b_{ij} = b_{ji} > 0$ for $i \neq j$. This system arises naturally when looking for standing wave solutions $\Psi_i(x,t) = e^{-i\lambda_i t} u_i(x)$ of the cubic nonlinear Schrödinger system

$$i\partial_t \Psi_i - \Delta \Psi_i = \mu_i |\Psi_i|^2 \Psi_i + \Psi_i \sum_{j\neq i} b_{ij} |\Psi_j|^2, \quad i = 1,\dots,d.$$

The parameters μ_i represent self-interactions within the same component, while b_{ij} ($i \neq j$) express the strength and the type of interaction between different components i and j. When $b_{ij} > 0$, this interaction is said to be of cooperative type, modeling phenomena appearing in nonlinear optics (see [15] and the physical references therein). On the other hand, a negative coefficient b_{ij} denotes competition, a feature arising, for instance, when modeling the Bose–Einstein condensation (see for instance [18]).

The assumption $b_{ij} = b_{ji}$, which translates the fact that the interactions between components are symmetric, implies that the system is of gradient type, and solutions of (1.1) correspond then to the critical points of the C^2-action functional $I_d : (H^1(\mathbb{R}^N))^d \to \mathbb{R}$ defined by

$$I_d(\mathbf{u}) = I_d(u_1,\dots,u_d) := \frac{1}{2}\sum_{i=1}^d \|u_i\|_{\lambda_i}^2 - \frac{1}{4}\sum_{i=1}^d \mu_i |u_i|_4^4 - \frac{1}{2}\sum_{\substack{i,j=1 \\ i<j}}^d b_{ij}|u_i u_j|_2^2, \tag{1.2}$$

where

$$\|v\|_{\lambda_i}^2 := \int |\nabla v|^2 + \lambda_i \int v^2,$$

and $|\cdot|_p$ stands for the usual L^p-norm, $1 \leq p \leq \infty$.