Journal of Functional Analysis 271 (2016) 2293–2307

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Carl's inequality for quasi-Banach spaces

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A R T I C L E I N F O

Article history:
Received 15 December 2015
Accepted 21 May 2016
Available online 27 May 2016
Communicated by G. Schechtman

Keywords:
Gelfand numbers
Entropy numbers
Carl's inequality
Compressed sensing

A B S T R A C T

We prove that for any two quasi-Banach spaces X and Y and any $\alpha > 0$ there exists a constant $\gamma_\alpha > 0$ such that

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le \gamma_\alpha \sup_{1\le k\le n} k^\alpha c_k(T)$$

holds for all linear and bounded operators $T : X \to Y$. Here $e_k(T)$ is the k-th entropy number of T and $c_k(T)$ is the k-th Gelfand number of T. For Banach spaces X and Y this inequality is widely used and well-known as Carl's inequality. For general quasi-Banach spaces it is a new result.

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1 This author was supported by the DFG Research Center Matheon "Mathematics for key technologies" in Berlin, project A23.
2 This author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education and by the Neuron Fund for Support of Science.

http://dx.doi.org/10.1016/j.jfa.2016.05.011
0022-1236/

1. Introduction

The theory of s-numbers [7,30,32] (sometimes also called n-widths) emerged from the studies of geometry of Banach spaces and of operators between them but found many applications in numerical analysis as well as linear and non-linear approximation theory [9–11,29,27]. It turned out to be also useful in estimates of eigenvalues of operators [5,8, 25,31].

One of the most useful tools in the study of s-numbers is Carl's inequality [5], which relates the behavior of several of the most important scales of s-numbers to their entropy numbers (see below for the exact definitions). If X and Y are Banach spaces and if $T : X \to Y$ is a bounded linear operator between them, then Carl's inequality states that for every natural number $n \in \mathbb{N}$

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le \gamma_\alpha \sup_{1\le k\le n} k^\alpha s_k(T). \tag{1.1}$$

Here, $e_k(T)$ denotes the entropy numbers of T and $s_k(T)$ stands for any of the approximation, Gelfand, or Kolmogorov numbers. For the definition of these quantities, let $T : X \to Y$ be a bounded linear operator between quasi-Banach spaces X and Y. Then we define the Gelfand numbers $c_n(T)$, the Kolmogorov numbers $d_n(T)$, the approximation numbers $a_n(T)$ and the entropy numbers $e_n(T)$, respectively, by

$$\begin{aligned}
c_n(T) &= \inf_{\substack{M\subset X\\ \operatorname{codim} M<n}} \sup_{\substack{x\in M\\ \|x\|_X\le 1}} \|Tx\|_Y\\
d_n(T) &= \inf_{\substack{N\subset Y\\ \dim N<n}} \sup_{\|x\|_X\le 1} \inf_{z\in N} \|Tx - z\|_Y\\
a_n(T) &= \inf\{\|T-L\| : L : X\to Y, \operatorname{rank}(L) < n\}\\
e_n(T) &= \inf\Big\{\varepsilon > 0 : T(B_X) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_Y)\Big\}.
\end{aligned}$$

In the last definition, B_X can denote either the open or the closed unit ball in X. While usually the closed unit ball is used, for technical reasons we prefer to work with the open unit ball $B_X = \{x \in X : \|x\|_X < 1\}$.

The main result of this note is that Carl's inequality holds also for quasi-Banach spaces and Gelfand numbers.

Theorem 1.1. *Let X and Y be quasi-Banach spaces. Then for any $\alpha > 0$ there exists a constant $\gamma_\alpha > 0$ such that*

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le \gamma_\alpha \sup_{1\le k\le n} k^\alpha c_k(T) \tag{1.2}$$

holds for all linear and bounded operators $T : X \to Y$.