Journal of Functional Analysis 271 (2016) 2339–2358

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Local smoothing estimates for the massless Dirac–Coulomb equation in 2 and 3 dimensions

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ARTICLE INFO

Article history:
Received 23 December 2015
Accepted 5 April 2016
Available online 20 April 2016
Communicated by B. Schlein

Keywords:
Dispersive PDEs
Dirac equation
Smoothing estimates

ABSTRACT

We prove local smoothing estimates for the massless Dirac equation with a Coulomb potential in 2 and 3 dimensions. Our strategy is inspired by [9] and relies on partial wave subspaces decomposition and spectral analysis of the Dirac–Coulomb operator.

1. Introduction and generalities

The massless Dirac equation with an electric Coulomb potential reads

$$\begin{cases} iu_t + \mathcal{D}_n u - \dfrac{\nu}{|x|} u = 0, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{C}^N, \\ u(0,x) = f(x) \end{cases} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jfa.2016.04.003
0022-1236/

where the massless Dirac operator $\mathcal{D}_n$ is defined in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

as

$$\mathcal{D}_2 = -i(\sigma_1 \partial_x + \sigma_2 \partial_y) = \begin{pmatrix} 0 & -i\partial_z \\ -i\partial_{\bar{z}} & 0 \end{pmatrix}$$

(we denote $\partial_z = \partial_x - i\partial_y$ and $\partial_{\bar{z}} = \partial_x + i\partial_y$) in dimension $n = 2$ with $N = 2$, and

$$\mathcal{D}_3 = -i \sum_{k=1}^{3} \alpha_k \partial_k = -i(\alpha \cdot \nabla)$$

where the 4×4 Dirac matrices are given by

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{1.3}$$

in dimension $n = 3$, with $N = 4$.

The Dirac equation is widely used in physics to describe relativistic particles of spin 1/2 (see e.g. [27,37]). In particular, the $2D$ massless equation is used as a model for charge carriers in graphene, a layer of carbon atoms arranged in a honeycomb lattice (see e.g. the physics survey [31] and the mathematical papers [21,24,29]). Note that there have been many rigorous results in the recent years on the stationary (massive) Dirac equation, essentially in dimension 3 (see e.g. [17] for references). Comparatively, its dynamics has been less investigated up to now.

The σ_j and α_j matrices were introduced in view of making the Dirac operator a square root of the Laplace operator: therefore, they satisfy by construction the following anticommutating relations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{ik} \mathbb{I}_2, \quad j, k = 1, 2;$$
$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{ik} \mathbb{I}_4, \quad j, k = 1, 2, 3$$

(we recall that the spectrum of $\mathcal{D}_n$ is the whole line $\mathbb{R}$). These conditions ensure that

$$(i\partial_t - \mathcal{D}_n)(i\partial_t + \mathcal{D}_n) = (\Delta - \partial_{tt}^2) I_N,$$

which strictly links the dynamics of the *free* massless Dirac equation to a system of N decoupled wave equations. Therefore, dynamical properties for the free Dirac equation can directly be derived from their wave counterparts: dispersive properties of the flow can be mainly encoded in the celebrated family of *Strichartz estimates*, which are given by