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Continuous framings for Banach spaces

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A B S T R A C T

The theory of discrete and continuous frames was introduced for the purpose of analyzing and reconstructing signals mainly in Hilbert spaces. However, in many interesting applications the analyzed space is usually a Banach space, and consequently the stable analysis/reconstruction schemes need to be investigated for general Banach spaces. Parallel to discrete Hilbert space frames, the theory of atomic decompositions, p-frames and framings have been introduced in the literature to address this problem. In this paper we focus on continuous frames and continuous framings (alternatively, integral reconstructions) for Banach spaces by the means of g-Köthe function spaces, in which the involved measure space is σ-finite, positive and complete. Necessary and sufficient conditions for a measurable function to be an L_ρ-frame are obtained, and we obtain a decomposition result for the analysis operators of continuous frames in terms of simple Köthe–Bochner operators. As a byproduct we show that a Riesz type continuous frame doesn't exist unless the measure space is purely atomic. One of our main results shows that there is an intrinsic connection between continuous framings and g-Köthe function spaces.

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1. Introduction

The concept of discrete frames was first introduced by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [7] by Daubechies, Grossman and Meyer in 1986, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. A discrete frame is a countable family of elements in a separable Hilbert space which allows a stable decomposition of an arbitrary element into an expansion of the frame elements. For more details and some recent developments on discrete frames, we refer to [5,15,16].

One generalization of discrete frames was first proposed by Gröchenig [14] in 1991. This generalized frame is called 'X_d-frame' for Banach spaces, where X_d is a Banach space consisting of some scalar valued sequences. Let X be a Banach space with dual X^*, and $\{\varphi_i\}_{i=1}^{\infty}$ be a sequence in X^*. A central question is whether we can find a sequence $\{x_i\}_{i=1}^{\infty}$ in X such that the *reconstruction formula*

$$x = \sum_{i=1}^{\infty} \varphi_i(x) x_i$$

holds for every $x \in X$. The terminologies 'atomic decompositions' for operators and 'framings' were introduced in many literature to express such expansions, and some detailed analysis by using X_d-frames can be found for example in [3,4,6,16,26]. It is worth to point out that a (discrete) framing for a Hilbert space doesn't have to be a Hilbert space frame, and it is a concept that truly captures the Banach space nature for the decomposition and reconstruction scheme (see [4,16–18]).

Another generalization of discrete frames was proposed by Kaiser [19], and independently by Ali, Antoine and Gazeau [1]. They generalized the index set from a countable set to some measure space, and the frames in this case are known as 'continuous frames' [1,12], or 'frames associated with measurable spaces' [13], or 'generalized frames' [2], or 'coherent states' [1]. We remark that for this class of frames, the analyzed spaces are still Hilbert spaces.

The two generalizations above lead us to some natural questions. For example: *What are the continuous versions for 'X_d-frames' and 'framings' in Banach spaces? How are they related?* We will address these problems and related areas of interest in this paper.

Let X be a Banach space, X_d be a Banach space consisting of some scalar valued sequences indexed by a discrete set I, and (Ω, Σ, μ) be a σ-finite, positive measure space. Then the analysis operator of an X_d-frame for X is a bounded linear operator from X into X_d, and the analysis operator of a continuous frame for a Hilbert space X is a bounded linear operator from X into $L_2(\Omega, \Sigma, \mu)$, where $L_2(\Omega, \Sigma, \mu)$ serves as the 'synthesis space' of the frame. To define a 'continuous X_d-frame', we need to find a more general function space to substitute the synthesis space $L_2(\Omega, \Sigma, \mu)$. In this paper we propose to use the Köthe space as a candidate for the synthesis space, which has been