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Quantum singular complete integrability [☆]

Thierry Paul [a,*], Laurent Stolovitch [b]

[a] CMLS, Ecole Polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau Cedex, France
[b] CNRS and Laboratoire J.-A. Dieudonné, Université de Nice, Sophia Antipolis Parc Valrose, 06108 Nice Cedex 02, France

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We consider some perturbations of a family of pairwise commuting linear quantum Hamiltonians on the torus with possibly dense pure point spectra. We prove that their Rayleigh–Schrödinger perturbation series converge near each unperturbed eigenvalue under the form of a convergent quantum Birkhoff normal form. Moreover the family is jointly diagonalized by a common unitary operator explicitly constructed by a Newton type algorithm. This leads to the fact that the spectra of the family remain pure point. The results are uniform in the Planck constant near $\hbar = 0$. The unperturbed frequencies satisfy a small divisors condition and we explicitly estimate how this condition can be released when the family tends to the unperturbed one. In the case where the number of operators is equal to the number of degrees of freedom – i.e. full integrability – our construction provides convergent normal forms for general perturbations of linear systems.

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* Corresponding author.
E-mail addresses: thierry.paul@polytechnique.edu (T. Paul), Laurent.STOLOVITCH@unice.fr (L. Stolovitch).

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1. Introduction

Perturbation theory belongs to the history of quantum mechanics, and even to its pre-history, as it was used before the works of Heisenberg and Schrödinger in 1925/1926. The goal at that time was to understand what should be the Bohr–Sommerfeld quantum conditions for systems nearly integrable [3], by quantizing the perturbation series provided by celestial mechanics [19]. After (or rather during its establishment) the functional analysis point of view was settled for quantum mechanics, the "modern" perturbation theory took place, mostly by using the Neumann expansion of the perturbed resolvent, providing efficient and rigorous ways of establishing the validity of the Rayleigh–Schrödinger expansion and leading to great success of this method, in particular the convergence under a simple argument of size of the perturbation in the topology of operators on Hilbert spaces [16], and Borel summability for (some) unbounded perturbations [13,24]. On the other hand, by relying on the comparison between the size of the perturbation and the distance between consecutive unperturbed eigenvalues, the method has two drawbacks: it remains local in the spectrum in the (usual in dimension larger than one) case of spectra accumulating at infinity and is even inefficient in the case of dense point unperturbed spectra which can be the case in the present article.

In the present article, we consider some commuting families of operators on $L^2(\mathbb{T}^d)$ close to a commuting family of unperturbed Hamiltonians whose spectra are pure point and might be dense for all values of $\hbar$. As already emphasized, standard (Neumann series expansion) perturbation theory does not apply in this context. Nevertheless, we prove that, under some asumptions, the pure point property is preserved and moreover, we show that the perturbed spectra are analytic functions of the unperturbed ones. All these results are obtained using a method inspired by classical local dynamics, namely the analysis of quantum Birkhoff normal forms (QBNF). Let us first recall some known fact of (classical) Birkhoff normal forms.

In the framework of (classical) local dynamics, Rüssmann proved in [23] (see also [4] and [11]) the remarkable result which says that, when the Birkhoff normal form (BNF), at any order, depends only on the unperturbed Hamiltonian, then it converges provided that the small divisors of the unperturbed Hamiltonian do not accumulate the origin too fast (we refer to [1] for an introduction to this subject). This leads to the integrability of the perturbed system. On the other hand, Vey proved two theorems about the holomorphic normalization of families of $l-1$ (resp. l) of commuting germs of holomorphic vector fields, volume preserving (resp. Hamiltonian) in a neighborhood of the origin of $\mathbb{C}^l$ (resp. $\mathbb{C}^{2l}$) (and vanishing at the origin) with diagonal and independent 1-jets [28,29].

These results were extended by one of us in [25,26] (see also [27]), in the framework of general local dynamics of a families of $1 \leqslant m \leqslant l$ commuting germs of holomorphic vector fields near a fixed point. It is proved that under an assumption on the formal (Poincaré) normal form of the family and under a generalized Brjuno type condition of