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On spectral stability of the nonlinear Dirac equation

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ABSTRACT

We study the point spectrum of the nonlinear Dirac equation in any spatial dimension, linearized at one of the solitary wave solutions. We prove that, in any dimension, the linearized equation has no embedded eigenvalues in the part of the essential spectrum beyond the embedded thresholds. We then prove that the birth of point eigenvalues with nonzero real part (the ones which lead to linear instability) from the essential spectrum is only possible from the embedded eigenvalues or thresholds, and therefore can not take place beyond the embedded thresholds. We also prove that "in the nonrelativistic limit" $\omega \to m$, the point eigenvalues can only accumulate to 0 and $\pm 2mi$.

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1. Introduction

In the present work, we study the discrete spectrum of linearization of nonlinear Dirac models. The analysis of the discrete spectrum is crucial for the question of the dynamical stability of solitary wave solutions. While this question is well-understood in many cases for both the nonlinear Schrödinger and Klein–Gordon equations (see e.g. the review [74]), there are numerous open questions for systems with Hamiltonians which are sign-indefinite, such as the nonlinear Dirac equation or the Dirac–Maxwell system.

The idea to consider self-interacting spinor field has been studied in Physics for a long time, starting with the pioneering work of Ivanenko [52] and then followed up in [39,40,48]. Widely known are the massive Thirring model [76] (spinor field with the vector self-interaction) and the Soler model [73] (spinor field with the scalar self-interaction). The one-dimensional analogue of the Soler model is known as the (massive) Gross–Neveu model [47,59].

In the past two decades there has been an increasing interest in the nonlinear Dirac equation. The bibliography is now so extensive that we do not hope to cover it comprehensively, only giving a very brief account. The existence of standing waves in the nonlinear Dirac equation was studied in [73,23,64,35]. Local and global well-posedness of the nonlinear Dirac equation was further addressed in [34] (semilinear Dirac equation in $(3+1)$D) and in [60] (nonlinear Dirac equation in $(3+1)$D). There are many results on the local and global well-posedness in $(1+1)$D; we mention [72,61,22,66,50].

The stability of solitary wave solutions of the nonlinear Dirac equation was approached via numerical simulations [70,3,4,10,65,79] and via heuristic arguments [14,63,75,13,28], based on the analysis of whether the energy functional is minimized under the charge constraint with respect to dilations and other families of perturbations. The spectrum of the linearization at solitary waves of the nonlinear Dirac equation in $(1+1)$D was computed in [24,10], suggesting the absence of eigenvalues with positive real part for linearizations at small amplitude solitary waves; we will say that such solitary waves are *spectrally stable*. The numerical simulations of the evolution of perturbed solitary waves [65] suggest that the small amplitude solitary waves in $(1+1)$D nonlinear Dirac equation are also *dynamically* stable (or *nonlinearly stable*). For the massive Thirring model, which is completely integrable, the orbital stability was proved by means of a coercive conservation law in [67,27]. The asymptotic stability of small amplitude solitary waves in the external potential has been studied in [17,18,68].

Given a real-valued function $f \in C(\mathbb{R})$, $f(0)=0$, we consider the following nonlinear Dirac equation in $\mathbb{R}^n$, $n \ge 1$, which is known as the Soler model [73]:

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n. \tag{1.1}$$

Above, $D_m = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m$ is the free Dirac operator. Here $\boldsymbol{\alpha} = (\alpha^\jmath)_{1\le \jmath\le n}$, with $\alpha^\jmath$ and β the self-adjoint $N \times N$ Dirac matrices (see Section 1.1 for the details); $m > 0$