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Metric selfduality and monotone vector fields on manifolds

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We develop a "metrically selfdual" variational calculus for c-monotone vector fields between general manifolds X and Y, where c is a coupling on $X \times Y$. Remarkably, many of the key properties of classical monotone operators known to hold in a linear context extend to this non-linear setting. This includes an integral representation of c-monotone vector fields in terms of c-convex selfdual Lagrangians, their characterization as a partial c-gradients of antisymmetric Hamiltonians, as well as the property that these vector fields are generically single-valued. We also use a symmetric Monge–Kantorovich transport to associate to any measurable map its closest possible c-monotone "rearrangement". We also explore how this metrically selfdual representation can lead to a global variational approach to the problem of inverting c-monotone maps, an approach that has proved efficient for resolving non-linear equations and evolutions driven by monotone vector fields in a Hilbertian setting.

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Contents

1. Introduction and main results

Many aspects of convexity theory such as Fenchel–Legendre duality, subdifferentiability, and cyclic monotonicity have been extended to settings where the usual linear duality $\langle x, x^* \rangle$ between a Banach space X and its dual X^* is replaced by a general coupling $c(x, y)$ of two arbitrary sets X and Y. These nonlinear "metric" generalizations of convexity and cyclic monotonicity were mostly motivated by problems in Riemannian geometry [24], mathematical economics [7,23], and by the Monge–Kantorovich theory of mass transport corresponding to general cost functions [29]. For example, McCann's extension of Brenier's theorem [4] to manifolds required that the scalar product in the linear theory be replaced by $c(x, y) = -d^2(x, y)$, where d is the Riemannian metric and where convexity is replaced by the concept of *"c-convexity"* described below. What is remarkable is that many of the key structural results known to hold under assumptions of classical convexity and cyclic monotonicity on Euclidean space extend to this metric setting. This has had major impact on non-linear analysis, differential geometry and their applications. See for example the books of Villani [29,30]. In this paper, we show that similar metric extensions hold for the notions of selfduality and for vector fields that are merely 2-monotone.

On the other hand, the most natural extensions of gradient flows of convex energies on a Hilbert space to curved manifolds seem to be those corresponding to *"pathwise convex"* and not necessarily *"metrically convex"* energies. Indeed, Otto's calculus [29] allows the rewrite of several non-linear evolution equations as gradient flows of geodesically convex free energy functionals on the Wasserstein manifold. Unfortunately, the global variational methods that characterize the success of convex analysis on linear spaces do not readily extend to non-linear settings. Instead, time discretization methods had to be used to circumvent the lack of global variational principles. See for example the penetrating study of Ambrosio–Gigli–Savaré [2], or the more recent comprehensive notes of Ambrosio–Gigli [1]. Our quest for global variational methods in nonlinear settings eventually led us to interesting connections between these two notions of convexity, including a new criterion to deduce the metric convexity of pathwise convex functionals, which normally is not easy to verify.