Journal of Functional Analysis 271 (2016) 245–263

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Gaps of consecutive eigenvalues of Laplace operator and the existence of multiple solutions for superlinear elliptic problem

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ARTICLE INFO

Article history:
Received 12 October 2014
Accepted 23 April 2016
Available online 5 May 2016
Communicated by H. Brezis

MSC:
35P05
35A15

Keywords:
New homological linking theorem
The gaps of consecutive eigenvalues
The fourth nontrivial solution
Ambrosetti–Rabinowitz condition

ABSTRACT

This paper is mainly related to multiple nontrivial solutions of the elliptic boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-2}\, u + f\,(x,u)\,, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

for $p \in (2, 2^*)$. It is reasonable to guess that for $\dim \Omega \geq 2$ above problem possesses infinitely many distinct solutions since this is proved to be true for ODE. However, so far one does not even know if there exists a fourth nontrivial solution. By using a new homological linking theorem, Morse theory, and some precise estimates we disclose the relationship among the gaps of consecutive eigenvalues of Laplace operator, growth trend of nonlinear terms and the existence of multiple solutions of superlinear elliptic boundary value problem. Moreover, as p is close to 2, we get the fourth nontrivial

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[1] The first author is supported by NSFC (11471319) and BCMIIS.
[2] The second author is supported by NSFC (11271353) and NSFC (11471319).

http://dx.doi.org/10.1016/j.jfa.2016.04.020

solution under appropriate hypotheses, where $f(x,u)$ satisfies Ambrosetti–Rabinowitz condition.

1. Introduction

Originating in the pioneer work of Ambrosetti–Rabinowitz [1] in 1973, followed by Rabinowitz [13], Benci–Rabinowitz [5], W.M. Ni [12] and others, linking methods have been powerful tools for finding saddle points of superquadratic functionals. In particular, from its first appearance in the work of Ambrosetti and Rabinowitz [1] the existence and multiplicity of solutions to the nonlinear elliptic equation:

$$\begin{cases} -\Delta u = au + h(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

Ω a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^N$, have been widely studied by many authors under the following hypotheses:

- (h_1) $h \in C(\Omega \times \mathbb{R}, \mathbb{R})$;
- (h_2) There are constants $C > 0$ and $p \in (2, 2^*)$ such that $|h(x,s)| \le C\left(1 + |s|^{p-1}\right)$ for $x \in \Omega$, $s \in \mathbb{R}$, where $2^* = \frac{2N}{N-2}$ for $N \ge 3$ and $2^* = \infty$ for $N = 1, 2$;
- (h_3) $h(x,s) = o(s)$ uniformly in x as $s \to 0$;
- (h_4) There are $\mu > 2$ and $M > 0$, such that for all $x \in \Omega$ and $|u| \ge M$,

$$0 < \mu H(x,u) \le uh(x,u),$$

where $H(x,u) = \int_0^u h(x,s)\,ds$.

More precisely, with the assumptions (h_1)–(h_4) Ambrosetti and Rabinowitz showed that as $a = 0$ the equation (1.1) admits at least two nontrivial solutions in [1]. Under the same hypotheses, further progress was achieved by Z.Q. Wang [20], who found the third one in 1991 by employing linking approach and Morse theory. On the other hand, it has been widely expected that an infinite number of solutions persists for the superlinear case without symmetric assumption (see [18, p. 124]). Indeed this is true for the case of ordinary differential equations, see Nehari [11], Struwe [16], Struwe [17], Struwe [18], Turner [19], Rabinowitz [14] and Rabinowitz [15]. However, to the best of our knowledge, for the case of partial differential equations, this conjecture has remained open over the past fifty years since Nehari's work in 1961 (see [18, p. 124] for details).

This paper is mainly concerned with constructing multiple nontrivial solutions of the elliptic boundary value problem: