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Optimal control of systems with noisy memory and BSDEs with Malliavin derivatives

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ABSTRACT

In this article we consider a stochastic optimal control problem where the dynamics of the state process, $X(t)$, is a controlled stochastic differential equation with jumps, delay and *noisy memory*. The term noisy memory is, to the best of our knowledge, new. By this we mean that the dynamics of $X(t)$ depend on $\int_{t-\delta}^{t} X(s)dB(s)$ (where $B(t)$ is a Brownian motion). Hence, the dependence is noisy because of the Brownian motion, and it involves memory due to the influence from the previous values of the state process.

We derive necessary and sufficient maximum principles for this stochastic control problem in two different ways, resulting in two sets of maximum principles. The first set of maximum principles is derived using Malliavin calculus techniques, while the second set comes from reduction to a discrete delay optimal control problem, and application of previously known results by Øksendal, Sulem and Zhang. The maximum principles also apply to the case where the controller has only

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partial information, in the sense that the admissible controls are adapted to a sub-σ-algebra of the natural filtration.

1. Introduction

In this article, we develop two approaches for analyzing optimal control for a new class of stochastic systems with noisy memory. The main objective is to derive necessary and sufficient criteria for maximizing the performance functional on the underlying set of admissible controls. One should note the following unique features of the analysis:

- The state dynamics follows a controlled stochastic differential equation (SDE) driven by *noisy memory*: The evolution of the state X at any time t is dependent on its past history $\int_{t-\delta}^{t} X(s)\,dB(s)$ where δ is the memory span and dB is white noise. In our opinion, it is reasonable and natural to consider this type of noisy dependence of the past.
- The maximization problem is solved through a new backward stochastic differential equation (BSDE) that involves not only partial derivatives of the Hamiltonian but also their Malliavin derivatives.
- Two independent approaches are adopted for deriving necessary and sufficient maximum principles for the stochastic control problem: The first approach is via Malliavin calculus and the second is a reduction of the dynamics to a two-dimensional controlled SDE with *discrete delay* and no noisy memory. In the second approach, the optimal control problem is then solved without resort to Malliavin calculus.
- A natural link between the above two approaches is established as we show that a solution of the noisy memory BSDE can be obtained from a solution of the two-dimensional (time-) advanced BSDE (ABSDE) and vice versa.
- To illustrate the usefulness of the Malliavin calculus approach, we outline in Section 8 an extension of the noisy memory problem where *the state dynamics cannot be reduced to a two-dimensional setting with discrete delay.*

To be somewhat more specific, we will outline below the scope of the results in the article. More precise regularity and measurability assumptions are provided in Sections 2, 3 and 4.

The dynamics is described by the following one-dimensional controlled stochastic functional differential equation with *noisy memory*:

$$dX(t) = b(t, X(t), Y(t), Z(t), \pi(t))dt + \sigma(t, X(t), Y(t), Z(t), \pi(t))\,dB(t) \tag{1.1}$$