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In between the inequalities of Sobolev and Hardy

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ABSTRACT

We establish both sufficient and necessary conditions for the validity of the so-called Hardy–Sobolev inequalities on open sets of the Euclidean space. These inequalities form a natural interpolating scale between the (weighted) Sobolev inequalities and the (weighted) Hardy inequalities. The Assouad dimension of the complement of the open set turns out to play an important role in both sufficient and necessary conditions.

1. Introduction

The *Sobolev inequality* is a fundamental tool in all analysis related to partial differential equations and variational problems, see e.g. [7,29]. When $G \subset \mathbb{R}^n$ is an open set and $1 \le p < n$, this inequality states that

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$$\left(\int_G |f|^{np/(n-p)}\,dx\right)^{(n-p)/np} \le C\left(\int_G |\nabla f|^p\,dx\right)^{1/p} \qquad \text{for all } f \in C_0^\infty(G)\,, \tag{1}$$

where the constant $C > 0$ depends only on n and p. If G is bounded (or of finite measure) and $1 \le q \le np/(n-p) =: p^*$, a simple use of Hölder's inequality yields a corresponding inequality where on the left-hand side of (1) the p^*-norm is replaced by the q-norm; the constant in the inequality then depends on the measure of G as well. In particular, for $q = p$ this gives the so-called *Friedrichs' inequality*

$$\left(\int_G |f|^p\,dx\right)^{1/p} \le C\left(\int_G |\nabla f|^p\,dx\right)^{1/p} \qquad \text{for all } f \in C_0^\infty(G)\,.$$

However, if $p > 1$ and the open set G satisfies some additional properties, e.g. G is a Lipschitz domain or more generally the complement of G is uniformly p-fat, then Friedrichs' inequality can be improved into a *p-Hardy inequality*

$$\left(\int_G |f|^p \delta_{\partial G}^{-p}\,dx\right)^{1/p} \le C\left(\int_G |\nabla f|^p\,dx\right)^{1/p} \qquad \text{for all } f \in C_0^\infty(G), \tag{2}$$

where $\delta_{\partial G}(x) = \operatorname{dist}(x, \partial G)$ denotes the distance from $x \in G$ to the boundary of G; see e.g. Lewis [26] and Wannebo [35]. Unlike Friedrichs' inequality, this p-Hardy inequality can be valid even if the open set G has infinite measure. A weighted (p, β)-Hardy inequality is obtained from inequality (2) by replacing dx with $\delta_{\partial G}^{\beta}\,dx$, $\beta \in \mathbb{R}$, on both sides of (2). Such an inequality holds, for instance, in a Lipschitz domain G for $1 < p < \infty$ if (and only if) $\beta < p - 1$, as was shown by Nečas [30]. On the other hand, if, roughly speaking, ∂G contains an isolated part of dimension $n - p + \beta$, then the (p, β)-Hardy inequality can not be valid in $G \subset \mathbb{R}^n$; we refer to [20,23].

In this paper, we are interested in certain inequalities forming a natural interpolating scale in between the (weighted) Sobolev inequalities and the (weighted) Hardy inequalities. More precisely, we say that an open set $G \subsetneq \mathbb{R}^n$ *admits a (q, p, β)-Hardy–Sobolev inequality* if there is a constant $C > 0$ such that the inequality

$$\left(\int_G |f|^q\,\delta_{\partial G}^{(q/p)(n-p+\beta)-n}\,dx\right)^{1/q} \le C\left(\int_G |\nabla f|^p\,\delta_{\partial G}^{\beta}\,dx\right)^{1/p} \tag{3}$$

holds for all $f \in C_0^\infty(G)$. Notice how the Sobolev inequality (1) is obtained as the case $q = p^* = np/(n-p)$, $\beta = 0$ in (3); and the weighted (p, β)-Hardy inequality is exactly the case $q = p$ in (3).

We begin in Section 2 by showing that if an open set $G \subset \mathbb{R}^n$ admits a (p, β)-Hardy inequality, then also (q, p, β)-Hardy–Sobolev inequalities hold for all $p \le q \le p^*$, see Theorem 2.1. Thus, for these q, sufficient conditions for Hardy inequalities always yield