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Dealing with moment measures via entropy and optimal transport

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ABSTRACT

A recent paper by Cordero-Erausquin and Klartag provides a characterization of the measures μ on $\mathbb{R}^d$ which can be expressed as the moment measures of suitable convex functions u, i.e. are of the form $(\nabla u)_\# e^{-u}$ for $u : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and finds the corresponding u by a variational method in the class of convex functions. Here we propose a purely optimal-transport-based method to retrieve the same result. The variational problem becomes the minimization of an entropy and a transport cost among densities ρ and the optimizer ρ turns out to be e^{-u}. This requires to develop some estimates and some semicontinuity results for the corresponding functionals which are natural in optimal transport. The notion of displacement convexity plays a crucial role in the characterization and uniqueness of the minimizers.

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1. Introduction

We consider in this paper the notion of *moment measure* of a convex function, which comes from functional analysis and convex geometry. Given a convex function $u : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we define its moment measure as

$$\mu := (\nabla u)_{\#}\rho, \quad \text{where } \mathrm{d}\rho = e^{-u(x)}\mathrm{d}x.$$

The connection of this notion with the theory of optimal transport is straightforward from the fact that, by Brenier's Theorem, the map ∇u will be the optimal transport map for the quadratic cost $c(x, y) = \frac{1}{2}|x - y|^2$ from ρ to μ.

In a recent paper, Cordero-Erausquin and Klartag ([9]) studied the conditions for a measure μ to be the moment measure of a convex function. First, they identified that an extra requirement has to be imposed to the function u in order the problem to be meaningful. The main difficulty arises in case u is infinite out of a proper convex set $K \subset \mathbb{R}^d$. In this case one needs to require some continuity properties of u on ∂K. Without this condition, every measure with finite first moment can be the moment of a function u, which is in general discontinuous on $\partial\{u < +\infty\}$. Also, without this condition there is a strong non-uniqueness of u. On the contrary, if one restricts to convex functions u that are continuous $\mathcal{H}^{d-1}$-a.e. on $\partial\{u < +\infty\}$ (those functions are called *essentially continuous*), then there is a clear characterization: a measure μ is a moment measure if and only if it has finite first moment, its barycenter is 0, and it is not supported on a hyperplane. Moreover, the function u is uniquely determined by μ up to space translations.

In [9], the authors first prove that these conditions on μ are necessary, due to summability properties of log-concave densities, and they prove that they are sufficient to build u as the solution of a certain minimization problem.

Here we want to reprove the same existence result with a different method, replacing functional inequalities techniques with ideas from optimal transport. This aspect seems to be absent from [9] even if it is not difficult to translate most of the ideas and techniques of Cordero-Erausquin and Klartag into their optimal transport counterparts. The result is an alternative language, that is likely to be appreciated by people knowing optimal transport theory, while the community of functional inequalities could legitimately prefer the original one (or, better, everyone could learn something from the language of the others). The question of which approach will the colleagues working with both optimal transport and functional inequality prefer is an open and unpredictable issue...

The main idea justifying this approach is the following: many variational problems of the form

$$\min \left\{ \frac{1}{2} W_2^2(\rho, \mu) + \int f(\rho(x))\,\mathrm{d}x \right\}$$