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Existence results for the fractional Nirenberg problem

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A B S T R A C T

We consider the fractional Nirenberg problem on the standard sphere $\mathbb{S}^n$ with $n \geq 4$. Using the theory of critical points at infinity, we establish an Euler–Hopf type formula and obtain some existence results for curvature satisfying assumptions of Bahri–Coron type.

1. Introduction

The famous Nirenberg problem in conformal geometry is: on the sphere $\mathbb{S}^n$ ($n \geq 2$) with standard metric g_0, is there a representation g of the conformal class $[g_0]$ such that g has scalar curvature (Gauss curvature for $n = 2$) equal to a prescribed function K? This problem is equivalent to the following equations

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$$-\Delta_{g_0} u + 1 = K e^u, \quad \text{on } \mathbb{S}^2,$$
$$-\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = K u^{\frac{n+2}{n-2}}, \quad \text{on } \mathbb{S}^n, \quad n \geq 3, \tag{1.1}$$

where R_g is the scalar curvature of g.

The linear operator on the left of (1.1) is known as the conformal Laplacian associated to the metric g_0 and is denoted as $P_1^{g_0}$. Another conformally covariant operator is

$$P_2^g = (-\Delta_g)^2 - \mathrm{div}_g(a_n R_g g + b_n Ric_g)d + \frac{n-4}{2} Q_n^g,$$

which was discovered by Paneitz, see [32] and [20]. Here Q_n^g, Ric_g are the standard Q-curvature and the Ricci curvature of g respectively, a_n, b_n are constants depending on n. P_1 and P_2 (with g being omitted when there is no ambiguity) are the first two terms of a sequence of conformally covariant elliptic operators $\{P_k\}$, which exists for all $k \in \mathbb{N}$ when n is odd, but only for $k \in \{1, \cdots, n/2\}$ when n is even. The first construction of these operators was by Graham, Jenne, Masion and Sparling in [24]. Thus a natural question is: are there any conformally covariant pseudodifferential operators of noninteger orders? In [33], the author constructed an intrinsically defined, arbitrary real number order, conformally covariant pseudo-differential operator. In the work of Graham and Zworski [25], it was showed that P_k can be realized as the residues at $\gamma = k$ of a meromorphic family of scattering operators. In this point of view, a family of conformally covariant pseudodifferential operators P_γ^g for noninteger γ was given.

In recent years, there are extensive works on the properties of the fractional Laplacian as non-local operators together with their applications to various problems, for example, [12,10,13,11,14] and so on. It is well known that $(-\Delta)^\gamma$ on $\mathbb{R}^n$ with $\gamma \in (0,1)$ is a nonlocal operator. In the remarkable work of Caffarelli and Silvestre [12], the authors express this nonlocal operator as a generalized Dirichlet–Neumann map for an elliptic boundary value problem with local differential operators defined on $\mathbb{R}_+^{n+1}$. And in the work of Chang and Gonzalez [15], the authors extended the work of [12] and characterized P_γ^g as such a Dirichlet-to-Neumann operator on a conformally compact Einstein manifold.

The operator P_γ^g with $\gamma \in (0, \frac{n}{2})$ has the following conformally covariant property: if $g = v^{\frac{4}{n-2\gamma}} g_0$, then

$$P_\gamma^{g_0}(vf) = v^{\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(f) \tag{1.2}$$

for any smooth function f, see [15]. Generalizing the formula for scalar curvature and the Paneitz Branson Q-curvature, the Q-curvature for g of order γ is defined as

$$Q_\gamma^g = P_\gamma^g(1).$$

In this paper, we are interested in the fractional Nirenberg problem on the standard sphere $\mathbb{S}^n$. That is to say, we want to find a representation g of the conformal class $[g_0]$